ELSEVIER International Journal of Approximate Reasoning 26 (2001) 1–46

INTERNATIONAL JOURNAL OF APPROXIMATE REASONING

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A commutative l-monoid for classification with fuzzy attributes

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Received November 1998; accepted June 2000

Abstract

In this paper the properties of an algebraic fuzzy structure are investigated in detail. The structure is suitable for modeling classifications through clusters composed of conventional sets and fuzzy attributes. We show that the structure is an integral commutative l-monoid. The expressive power of the structure is such that several situations can be viewed as classification problems, e.g., fuzzy assessment of students, user modeling for fuzzy hypermedia systems, spaces of the cognitive states of the user of a tutoring system, financial investments, medical diagnoses. The problem of getting the unknown classification beginning from the final classification is deeply investigated and it is shown that the problem is strictly related to the solution of an equation in the monoid. Thus it is possible to construct procedures of the type 'what happens if' which permit to attain significant results both on the theoretical side and the applicative one. Finally, by means of this approach, both the absolute and the relative relevance of an attribute are defined and evaluated, given a universe of discourse and a set of classifications. Moreover, this couple of features allow to develop a sophisticated analysis of how a new attribute can be obtained beginning from a set of attributes. © 2001 Elsevier Science Inc. All rights reserved.

Keywords: Fuzzy number; Monoid; Lattice; Classical partition; Relevance; Classification

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PII: S0888-613X(00)00053-0

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1. Introduction

In many situations in the field of scientific research and its applications the necessity arises, in order to solve a specific problem, of classifying the elements of a set with as to a given feature. As a consequence of this need, clustering algorithms have seen an explosive development over the past 20 years [1,2,9,11–16,23,25,27,28] and a large set of techniques and methods have been applied to data originating in various contexts. We just quote medicine (classification of diseases), archaeology (classifications during excavations), astronomy (clustering of galaxies), economy (clustering of firms), and so on. Clustering techniques distribute the elements of a set as to a given feature. For example, individuals in a population can be clustered according to their tallness.

Researchers in the field have been investigating mainly two classes of problems:

- given a classification of the universe of discourse, single out the rules to assign each element of the universe to a specific class;
- given a universe of discourse, classify its elements according to specific goals.

From the formal point of view, the problem can be specified as follows:

Let $I = \{a_1, a_2, a_3, \dots, a_n\}$ be our set and $X = [x_1, x_2, x_3, \dots, x_n]$ be the n-ple of assessments of the elements of I as to a specific feature C, i.e., x_i is the assessment of a_2 as regards C, and so on. The problem to be tackled is to single out, given the n-ple X, clusters that represent a partition of I. Moreover elements belonging to the same subset should be classified as *similar* as regards C, whereas elements belonging to different subsets are classified as *different*.

It is also possible to widen the problem by introducing an *optimality criterion*, e.g., a function denoting the levels of desirability of possible partitions [11].

In clustering problems involving quantitative data a suitable *distance* (*measure of dissimilarity*) or an *affinity index* (*measure of similarity*) can be introduced to cope with the values arising during measurements [11].

In general, clustering algorithms are based on these two techniques:

(a) single out either the partition that minimizes the dispersion within each cluster(1975) or that maximizes the dispersion among clusters (*variance-based partition algorithms*);

(b) construct hierarchical-like clusters, in which data are organized according to a nested sequence of groups (*hierarchical algorithms*).

The methods so far sketched out require that each element belong to one and only one cluster, thus they are based on numerical quantities and relative numerical manipulation. In fact we have that the situation is rather vague, the elements are each other 'nearby' and some of them are equally likely to belong to more than one cluster. Consequently one has situation in which clusters 'fade' each other. This kind of problems can be tackled by means of a

Two elements a, b belonging to $(I, \geqslant)$ are said *not comparable* if neither of the relations $a \geqslant b$, $b \geqslant a$ holds.

We say that $(I, \geqslant)$ is a *chain* if there is no pair of not comparable elements. If in $(I, \geqslant)$ there is at least one pair of not comparable elements, then $(I, \geqslant)$ is said *partially ordered*. An ordered set is said *complete* if every not empty part is has the l.u.b. and the g.l.b.

An algebraic structure $(S, \cap, \cup)$ is said *lattice* if the operations $\cup$ and $\cap$ are both commutative and associative and, moreover, if, for each pair of elements a and b one has $a \cap (a \cup b) = a$ and $a \cup (a \cap b) = b$.

Let R be a relation on $(S, \cap, \cup)$ such that aRb if and only if $a \cap b = a$ (or, equivalently $a \cup b = b$). Then R is reflexive, symmetric and transitive, thus it is an ordering relation on S. We denote R by $\leqslant$, then the ordered set $(S, \leqslant)$ is said *associated* to the reticle $(S, \cap, \cup)$.

In a similar way, if one has the ordered set $(S, \leqslant)$ such that for every pair a, b of S there are both the l.u.b. and the g.l.b. then one can introduce the following operations:

$$\cap : (a, b) \in S \times S \rightarrow \text{g.l.b.}\{a, b\}, \quad \cup : (a, b) \in S \times S \rightarrow \text{l.u.b.}\{a, b\}$$

thus the algebraic structure $(S, \cap, \cup)$ is a lattice and it is said *associated* to $(S, \leqslant)$. As one has that $a \cap b = a$ if and only if $a \leqslant b$ then the ordered set associated to the lattice $(S, \cap, \cup)$ is identical with $(S, \leqslant)$.

Let $(S, \cap, \cup)$ be a lattice, if there is a unit element with respect to $\cup$, such element is denoted by 0. As for each a one has $0 \cup a = a$, it follows $0 \leqslant a$ for each a, thus 0 is also the minimum of the ordered set $(S, \leqslant)$. In a similar way, if there is a unit element with respect to $\cap$, this element is denoted by 1 and since for each a one has $a \cap 1 = a (a \leqslant 1)$ then 1 is the maximum of the ordered set $(S, \leqslant)$. If the ordered set $(S, \leqslant)$ is endowed with both minimum and maximum elements the latter are, respectively, the unit element with respect to the operations $\cup$ and $\leqslant$ of the corresponding lattice.

A lattice is *complete* if such is the corresponding ordered set. It is apparent that a lattice with 0 and 1 is complete, as the corresponding ordered set is endowed with both the minimum and maximum. A lattice is modular if for every triple a, b, c such that $a \leqslant c$ one has

$$(a \cup b) \cap c = a \cup (b \cap c).$$

A lattice is said *distributive* if one of the previous conditions holds true. Every distributive lattice is modular.

Let S be a not empty set on which the associative operation $*$ is defined, then $(S, *)$ is a *semigroup*. If the operation $*$ has a unit element in S then $(S, *)$ is a *monoid*. In case the operation $*$ is commutative then $(S, *)$ is a monoid *commutative*.

Let $(S, \leqslant)$ be the lattice associated to the ordered set $(S, \leqslant)$. Suppose that S is endowed with 0 and 1, where $0 \neq 1$. Moreover, let $(S, *)$ be a commutative fuzzy-based approach [28]: each element is given a grade of membership for each cluster, so that each cluster becomes a fuzzy set. This concept can be formalized as follows. The resulting fuzzy clustering is a natural extension of the traditional one. Of course, the use of this kind of approach is suitable mainly for those data which are to be classified according to qualitative features.

In our approach we have the values of one or more linguistic variables [30] that generate partitions of the universe of discourse. An operation with two components very similar to the number-theoretic addition and multiplication is defined on sequences of couples (equivalence class, linguistic value) [17–21]. This operation induces a different classification, in which the list of couples (equivalence class, linguistic value) gets determined by the resulting classifications. Thus one has that the representation of data allows to represent the typical vagueness of a set of data arising in a real context but the processing of these data can be carried out via numerical computations, by using fuzzy-based techniques.

This paper is organized as follows. Section 2 presents basic results about posets, lattices and monoids. Section 3 discusses the meaning of the term 'classification'. The operation of composition of two or more classifications is illustrated in Section 4. Then a total order relation is introduced into the set of classifications and it is proved that the algebraic structure is an integral commutative l-monoid [24]. Section 6 discusses how the relevance can be computed. Section 7 presents a case study originally discussed in [28], the results attained by us are equal to Sato's but the analysis of the relevance permits to draw further information from the classifications, not obtainable in Sato's approach. In Section 8 an operation in some way similar to the classical division is introduced, its utility is apparent in Section 9 where it is shown how it is possible to tackle problems of the type 'if C is the classification obtained beginning from A and B, how A is to be modified in order to get a different classification C?' In our case study, we see that singling out A induces the possibility of developing a plan of preventive medicine in order to fight rachitism and other diseases typical of the age of puberty. In Section 9 additional computations are carried out on the data of our case study and finally the last section presents a synopsis of the theoretical results illustrated in the paper.

2. Mathematical preliminaries

Let $\geqslant$ be a binary relation on the set I. We say that $\geqslant$ is an *ordering on I* if the following properties hold true:

1. *Reflexive property*: $a \geqslant a$ for each $a \in I$.
2. *Asymmetric property*: given a, b in I, if $a \geqslant b$ and $b \geqslant a$, then $a = b$.
3. *Transitive property*: given a, b, c in I, if $a \geqslant b$ and $b \geqslant c$, then $a \geqslant c$.

Then the set I is said *ordered by* $\geqslant$ and we write $(I, \geqslant)$. In an ordered set, if $a \geqslant b$ and $a \neq b$ then we write $a > b$.

monoid, then the triple $(S, \leqslant, *)$ is said *commutative l-monoid.* $(S, \leqslant, *)$ is said *integral* if the 1 of the lattice $(S, \leqslant)$ is identical with the unit element of the monoid $(S, *)$.

Given a classical non empty set U, said the *universe of discourse*, a *fuzzy set* A in U is a function from U to [0,1]: A: $U \to [0, 1]$. This function if often denoted by $\mu_A(x)$. The definitions introduced by Zadeh [29] for equality, inclusion, union, intersection and complement are a suitable extension of the classical set-theoretic notions.

A fuzzy set is said *convex* if and only if:
for every $x_1, x_2 \in U$, and for every $p \in [0, 1]$
$\mu_A(px_1 + (1 - p)x_2) \geqslant \min[\mu_A(x_1), \mu_A(x_2)]$

A fuzzy set is *normalized* if there is an element for which the membership function takes value 1. A *fuzzy number* A is a fuzzy set on the set of real numbers such that.

1. A is convex;
2. there is exactly one $x_0 \in R$ such that $\mu_A(x_0) = 1$;
3. μ_A is stepwise continuous.

The membership function of a *triangular fuzzy number* is characterized by three numbers $l \leqslant c \leqslant r$ as one has

$$\mu_M(x) = \begin{cases} (x - l)/(c - l) & \text{if } x \in [l, c], \\ (r - x)/(r - c) & \text{if } x \in [c, r], \\ 0 & \text{otherwise.} \end{cases}$$

The importance of fuzzy numbers stems from the circumstance that Zadeh's principle of extension establishes a fruitful link with the traditional theory of numbers. In fact, via the principle of extension, the operations on real numbers can be extended to fuzzy sets. For example if $[a, b, c]$ and $[a', b', c']$ are two triangular numbers then $[a, b, c] + [a', b', c'] = [a + a', b + b', c + c']$, while the product of a triangular number $[a, b, c]$ by a scalar k gives $k * [a, b, c] = [k, k, k] * [a, b, c] = [k * a, k * b, k * c]$ [10] .

3. A suitable algebraic structure for classification

Let U be the universe of discourse. Suppose that each element of U is represented by the k-ples $(A1(u), \ldots, Ak(u))$, where the $Ai(u)$ are the fuzzy measures whose values are ordered triangular fuzzy numbers in [0, 1]. These fuzzy numbers represent the truth grade of the element u as regards the ith attribute; they can be represented by linguistic labels, such "almost true", "false" and so on. In turn, each attribute is represented by an *attribute string* as follows:

$$Aj : a_n^{\alpha_n} a_{n-1}^{\alpha_{n-1}} \ldots a_2^{\alpha_2} a_1^{\alpha_1},$$

where $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, are the possible values of the linguistic labels, and $a_j = A_j^{-1}(\alpha_j)$ are the subsets of U for which the jth attribute takes the value α_j.

Thus an attribute string is a string that can be interpreted as a classification induced by a fuzzy attribute given the universe of discourse. In the following, for the sake of simplicity and without loss of generality, the linguistic labels in the examples are just Completely True, True, Almost True, False, and the others are obtained by approximating these three labels. The following table reports the triangular numbers for each label:

F	false	$[0, 0, 0.2]$
AT	almost true	$[0.2, 0.4, 0.6]$
T	true	$[0.5, 0.7, 0.9]$
CT	completely true	$[0.8, 1, 1]$

Formally, attribute strings can be defined as follows:

An *attribute string* on a set U is a string

$$a_n^{\alpha_n} a_{n-1}^{\alpha_{n-1}} \ldots a_2^{\alpha_2} a_1^{\alpha_1}$$

and a list of positive integers $(k, d_n, d_{n-1}, \ldots, d_2, d_1)$ such that:
1. a_i is a subset of U, for each i;
2. $\{a_n, a_{n-1}, \ldots, a_2, a_1\}$ is a partition of U;
3. α_j is a triangular number in $[0, 1]$;
4. $\alpha_n > \alpha_{n-1} > \cdots > \alpha_2 > \alpha_1$.

Moreover the strings $U^{\mathrm{NI}}(1, 1)$ and $U^{\mathrm{NC}}(1, 1)$ are attribute strings.

In the following $S(U)$ denotes the set of attribute strings on the universe U.

In our approach $\{a_n, a_{n-1}, \ldots, a_2, a_1\}$ denotes the partition induced by the attribute, whereas the fuzzy numbers $\alpha_n, \alpha_{n-1}, \ldots, \alpha_2, \alpha_1$ correspond to the labels associated with each group of the partition. In the following $a_n, a_{n-1}, \ldots, a_2, a_1$ are denoted by the term *first parts* and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_2, \alpha_1$ by the term *second parts*. Strings having a list of numbers all equal to 1 are said *native strings* (for example, $\{1, 2, 5\}^{[0.2,0.5,0.7]}\{3, 4\}^{[0.1,0.3,0.5]}(1, 1, 1)$). The meaning of this terminology will be apparent in the following, when the composition of attribute strings is defined.

It is worth clarifying the meaning of the strings U^{NI} and U^{NC}. Suppose that our universe of discourse includes human beings and animals and we are interested in investigating the nature of a specific disease. Moreover, suppose that one of the attributes is 'high standard of living'. For the animals this attribute is meaningless and the label NC specifically denotes elements not compatible with the attribute.

Let us consider now the attribute 'high globular value' and suppose that for some individuals test have not been carried out. The label NI denotes those elements for which we lack any information about the attribute.

Thus, the set of labels is defined as follows: {NC, NI} ∪ {triangular numbers in [0, 1]}, provided that NI < f < NC, for all triangular numbers f in [0, 1].

The basic feature of NC is that it should not affect in any way the other labels. For example, given $U = \{a,b,c,d,e,f,g\}$, $A = [g,b]^{NC}[a]^{CT}[f,c]^{T}[e,d]^{AT}$ and $B = [g]^{NC}[a,b]^{T}[c,d]^{AT}[e,f]^{F}$, by composing A and B one expects that g keeps being not compatible, whereas B keeps taking the label T as in B. This happens because saying that g and b are not compatible with the attribute A means that A is 'unable' to single out g and b also in case B is taken into account. Thus, if C stems from the composition of A and B, we introduce the operation of composition so that

$$\mathrm{C}^{-1}(\mathrm{NC}) = \mathrm{A}^{-1}(\mathrm{NC}) \cap \mathrm{B}^{-1}(\mathrm{NC}).$$

In turn, as regards the label NI, if one has, for example,

$$A = [f,c]^{T}[e,d]^{AT}[g,b]^{F}[a]^{NI}, \quad B = [c,d]^{CT}[e,f]^{AT}[g]^{F}[a,b]^{NI},$$

we would expect that a and b keep taking the label NI also in C obtained from the composition of A and B. This is apparent for a, whereas for b the lack of information on A prevents from giving an assessment also in C. Thus also in this case, a constraint will be present

$$\mathrm{C}^{-1}(\mathrm{NI}) = \mathrm{A}^{-1}(\mathrm{NI}) \cup \mathrm{B}^{-1}(\mathrm{NI}).$$

The operation of composition introduced in the next section takes into account these aspects. Finally it is possible to show that the associative property holds for this operation if and only if consider the strings U^{NC} and U^{NI}. Consequently U^{NC} becomes the unit element and $\langle S(U), \diamondsuit, U^{NC}\rangle$ takes the structure of monoid.

We note that the sequence $(k, d_n, d_{n-1}, \ldots, d_2, d_1)$ contains this information about the related string [4]:

(i) the value k denotes the number of strings necessary for its generation,
(ii) the values d_i denote the number of clusters whose composition has generated the ith cluster, i.e., the ith element of the partition present in the string.

Example. Suppose that we have a set of individuals $U = \{a,b,c,d,e,f,g\}$, then the classification of such set with respect to the attribute Tallness may be represented by the following string:

$$\{a,c,e\}^{T}\{b,d\}^{AT}\{f,g\}^{F}(1,1,1,1).$$

In case the tallness of the individuals is as follows (in meters):

a 1.65, b 1.75, c 1.90, d 1.80, e 1.75, f 1.77, g 1.85

the classification induced by the attribute may be as follows:

$$\text{Tallness} = \{c, g\}^{\text{CT}}\{d\}^{\text{T}}\{b, e, f\}^{\text{AT}}\{a\}^{\text{F}}(1, 1, 1, 1, 1).$$

Other examples concerning a large variety of situations can be found in the following papers:

[19]	classifying some bacteria, regarding how dangerous they are to humans
[17]	approximate reasoning
[19]	generation of automatic diagnoses
[18]	diets for breast-feeding women
[7]	tutorial assessment
[20]	financials investments
[5]	user modeling in adaptive hypermedia systems
[8]	uncertainty processing in user-modeling activity

3.1. Linguistic approximation

Of course not all fuzzy numbers in [0, 1] can have a corresponding linguistic label, thus, in order that each string could be represented in not numerical form, it is necessary approximate the fuzzy values by means of linguistic labels.

Let $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ be the fuzzy numbers associated to some linguistic labels (in the previous examples $k = 4$) and let β be the fuzzy number to be approximated.

Suppose that the middle value of β (say m) lies between the middle values of α_i and α_{i+1}, i.e., belongs to the interval $[m_i, m_{i+1}]$ (for some $i = 1, \ldots, k$).

Let the quantity $m_{i+1} - m_i$ be denoted by d, then we can carry out the following approximation:

(i) if $m \in \{m_i, m_i + d/10\}$, then β is approximated by α_i;

(ii) if $m \in \{m_i + (d/10), m_i + (3/10) * d\}$ then β is said "next to" α_i and we write nt$[\alpha_i]$;

(iii) if $m \in \{m_i + (3/10) * d, m_i + (7/10) * d\}$ then β is "included between" α_i and α_{i+1} and we write ib$[\alpha_i, \alpha_{i+1}]$;

(iv) if $m \in \{m_i + (7/10) * d, m_i + (9/10) * d\}$ then β is "before" α_{i+1} and we write bt$[\alpha_{i+1}]$;

(v) if $m \in \{m_i + (9/10) * d, m_{i+1}\}$ then β is approximated by α_{i+1}.

We note that our rule of approximation gives an upper bound to the number of possible labels, in fact this number cannot exceed $4n - 3$ where n is the

Example. Let [0.3, 0.75, 0.8] be a triangular number, its linguistic approximation is nt[V], in fact T ≈ [0.5, 0.7, 0.9] and CT ≈ [0.8, 1, 1] then d = (1 − 0.7) = 0.3, thus $(0.7 + 0.3/10) = 0.73 < 0.75 < 0.79 = (0.7 + 0.3 * 3/10)$.

The following fuzzy numbers are approximated in a similar way:

$$\begin{array}{ll} [0.25, 0.35, 0.57] \approx \mathrm{nt[AT]}, & [0.03, 0.1, 0.4] \approx \mathrm{nt[F]}, \\ [0.01, 0.02, 0.08] \approx \mathrm{F}, & [0.01, 0.02, 0.9] \approx \mathrm{F}, \\ [0.74, 0.82, 0.95] \approx \mathrm{ib[T, CT]}, & [0.41, 0.63, 0.7] \approx \mathrm{b[T]}. \end{array}$$

4. Composing strings as a tool to classify

In order to get a classification tool, it is necessary to introduce an operation between the ordered strings associated with the attributes in such a way as to obtain a new string which is a finer classification of the information contained in the original strings.

The idea of such an operation can be intuitively understood if we consider the twofold meaning of a digit in a number, i.e., the roles of value and position. In the case of our strings, the absolute value of the generic element a_i is the set of the elements of U which a_i represents, whereas the value corresponding to the position is given by the linguistic label α_i.

Because of the heterogeneous structure of the string, it is necessary to distinguish the operations according to the two parts of the strings: one operation for the part concerning the subsets of U, and a different one applied to the part where fuzzy sets intervene.

Definition. The operation of binary composition of attribute strings $\Diamond : S(U) \times S(U) \to S(U)$ is defined as follows:

If $A = a_n^{\alpha_n} a_{n-1}^{\alpha_{n-1}} \ldots a_2^{\alpha_2} a_1^{\alpha_1} (k_A, d_{A,n}, \ldots, d_{A,2}, d_{A,1})$ and $B = b_m^{\beta_m} b_{m-1}^{\beta_{m-1}} \ldots b_2^{\beta_2} b_1^{\beta_1} (k_B, d_{B,n}, \ldots, d_{B,2}, d_{B,1})$ are strings then:

$$C = A \Diamond B = c_{m+n-1}^{\gamma_{m+n-1}} \ldots c_2^{\gamma_2} c_1^{\gamma_1} \; (k_A + k_B, d_{C,m+n1}, \ldots, d_{C,2}, d_{C,1}),$$

where

for $n \geqslant m$

as regards first parts one has:

$$C_i = \begin{cases} \bigcup_{j=1,\ldots,i} & a_{i-j+1} \cap b_j, \quad 1 \leqslant i \leqslant m-1, \\ \bigcup_{j=1,\ldots,m} & a_{i-j+1} \cap b_j, \quad m \leqslant i \leqslant n-1, \\ \bigcup_{j=i-n+1,\ldots,m} & a_{i-j+1} \cap b_j, \quad n \leqslant i \leqslant m+n-1 \end{cases}$$

for the second parts:

$$\gamma_i = \begin{cases} 1/((kA+kB)*dc,i)\sum_{j=1,\dots,i} d_{B,j}*d_{A,i-j+1}*(k_A*\alpha_{i-j+1}+k_B*\beta_j), \\ \quad 1\leqslant i\leqslant m-1, \\ 1/((kA+kB)*dc,i)\sum_{j=1,\dots,m} d_{B,j}*d_{A,i-j+1}*(k_A*\alpha_{i-j+1}+k_B*\beta_j), \\ \quad m\leqslant i\leqslant n-1, \\ 1/((kA+kB)*dc,i)\sum_{j=i-n+1,\dots,m} d_{B,j}*d_{A,i-j+1}*(k_A*\alpha_{i-j+1}+k_B*\beta_j), \\ \quad n\leqslant i\leqslant m+n-1, \end{cases}$$

where

$$d_{C,i} = \begin{cases} \sum_{j=1,\dots,i}(d_{B,j}*d_{A,i+1-j}), & 1\leqslant i\leqslant m-1, \\ \sum_{j=1,\dots,m}(d_{B,j}*d_{A,i+1-j}), & m\leqslant i\leqslant n-1, \\ \sum_{j=i-n+1,\dots,m}(d_{B,j}*d_{A,i+1-j}), & n\leqslant i\leqslant m+n-1 \end{cases}$$

for $n\leqslant m$

for the first parts one has:

$$c_i = \begin{cases} \bigcup_{j=1,\dots,i} & a_{i-j+1}\cap b_j, \quad 1\leqslant i\leqslant n-1, \\ \bigcup_{j=i-n+1,\dots,i} & a_{i-j+1}\cap b_j, \quad n\leqslant i\leqslant m-1, \\ \bigcup_{j=i-n+1,\dots,m} & a_{i-j+1}\cap b_j, \quad m\leqslant i\leqslant m+n-1 \end{cases}$$

for the second parts:

$$\gamma_i = \begin{cases} 1/((kA+kB)*dc,i)\sum_{j=1,\dots,i} d_{B,j}*d_{A,i-j+1}*(k_A*\alpha_{i-j+1}+k_B*\beta_j), \\ \quad 1\leqslant i\leqslant n-1, \\ 1/((kA+kB)*dc,i)\sum_{j=i-n+1,\dots,i} d_{B,j}*d_{A,i-j+1}*(k_A*\alpha_{i-j+1}+k_B*\beta_j), \\ \quad n\leqslant i\leqslant m-1, \\ 1/((kA+kB)*dc,i)\sum_{j=i-n+1,\dots,m} d_{B,j}*d_{A,i-j+1}*(k_A*\alpha_{i-j+1}+k_B*\beta_j), \\ \quad m\leqslant i\leqslant m+n-1 \end{cases}$$

where

$$d_{C,i} = \begin{cases} \sum_{j=1,\dots,i}(d_{B,j}*d_{A,i+1-j}), & 1\leqslant i\leqslant n-1, \\ \sum_{j=i-n+1,\dots,i}(d_{B,j}*d_{A,i+1-j}), & n\leqslant i\leqslant m-1, \\ \sum_{j=i-n+1,\dots,m}(d_{B,j}*d_{A,i+1-j}), & m\leqslant i\leqslant m+n-1. \end{cases}$$

Moreover, for each attribute string A one has $A\Diamond U^{\mathrm{NI}} = U^{\mathrm{NI}}\Diamond A = U^{\mathrm{NI}}$ and $A\Diamond U^{\mathrm{NC}} = U^{\mathrm{NC}}\Diamond A = A$.

This operation, at first look, appears very complex, anyway it is just the application of the algorithm for the multiplication of integers to the first parts, the second parts and the attribute strings. By composing two classifications, a third one arises where the information contained in the original classifications is synthetized. The initial strings, not generated via composition, are characterized by a sequence of 1, and thus they are called *native* strings.

Example. Suppose that one has the following strings concerning the attributes Tall and Fat:

$$\text{Tall} = \{c,g\}^{\text{CT}}\{d\}^{\text{T}}\{b,e,f\}^{\text{AT}}\{a\}^{\text{F}} \quad \text{and} \quad \text{Fat}\{g\}^{\text{CT}}\{a,d,e,f\}^{\text{AT}}\{b,c\}^{\text{F}}.$$

By composing the two strings one gets, for the first parts:

		{c,g}	{d}	{b,e,f}	{a}	⊗
			{g}	{a,d,e,f}	{b,c}	=
		{c}	∅	{b}	∅	
	∅	{d}	{e,f}	{a}	-	
{g}	∅	∅	∅	-		
{g}	∅	{c,d}	{e,f}	{a,b}	∅	

Thus Tall ◊ Fat induces a partition $\{\{g\},\{c,d\},\{e,f\},\{a,b\}\}$ on the universe of discourse. As regards the list of numbers, one has:

		1	1	1	1	®
			1	1	1	=
		1	1	1	1	
	1	1	1	1	–	
1	1	1	1	–		
1	2	3	3	2	1	

thus:

$$d_{\text{Tall}\Diamond\text{Fat},1} = 1, \quad d_{\text{Tall}\Diamond\text{Fat},2} = 2, \quad d_{\text{Tall}\Diamond\text{Fat},3} = 3, \quad d_{\text{Tall}\Diamond\text{Fat},4} = 3,$$

$$d_{\text{Tall}\Diamond\text{Fat},5} = 2, \quad d_{\text{Tall}\Diamond\text{Fat},6} = 1$$

and, moreover, one gets: $k_{\text{Tall}\Diamond\text{Fat}} = k_{\text{Tall}} + k_{\text{Fat}} = 1 + 1 = 2$.

In turn, as regards the second parts, one gets:

	[0.8,1,1]	[0.5,0.7,0.9]	[0.2,0.4,0.6]	[0,0,0.2]	©
		[0.8,1,1]	[0.2,0.4,0.6]	[0,0,0.2]	=
	$a_{4,1}$	$a_{3,1}$	$a_{2,1}$	$a_{1,1}$	
	$a_{4,2}$	$a_{3,2}$	$a_{2,2}$	$a_{1,2}$	
$a_{4,3}$	$a_{3,3}$	$a_{2,3}$	$a_{1,3}$		
γ_6	γ_5	γ_4	γ_3	γ_2	γ_1

where

$$a_{i,j} = d_{\text{Fat},j} * d_{\text{Tall},i-j+1} * (k_{\text{Tall}} * a_{i-j+1} + k_{\text{Fat}} * b_j)$$

$$\text{Tall} = \{c,g\}^{\text{CT}}\{d\}^{\text{T}}\{b,e,f\}^{\text{AT}}\{a\}^{\text{F}} \quad \text{and} \quad \text{Fat}\{g\}^{\text{CT}}\{a,d,e,f\}^{\text{AT}}\{b,c\}^{\text{F}}.$$

By composing the two strings one gets, for the first parts:

		{c,g}	{d}	{b,e,f}	{a}	⊗
			{g}	{a,d,e,f}	{b,c}	=
		{c}	∅	{b}	∅	
	∅	{d}	{e,f}	{a}	-	
{g}	∅	∅	∅	-		
{g}	∅	{c,d}	{e,f}	{a,b}	∅	

Thus Tall ◊ Fat induces a partition $\{\{g\},\{c,d\},\{e,f\},\{a,b\}\}$ on the universe of discourse. As regards the list of numbers, one has:

		1	1	1	1	®
			1	1	1	=
		1	1	1	1	
	1	1	1	1	–	
1	1	1	1	–		
1	2	3	3	2	1	

thus:

$$d_{\text{Tall}\Diamond\text{Fat},1} = 1, \quad d_{\text{Tall}\Diamond\text{Fat},2} = 2, \quad d_{\text{Tall}\Diamond\text{Fat},3} = 3, \quad d_{\text{Tall}\Diamond\text{Fat},4} = 3,$$

$$d_{\text{Tall}\Diamond\text{Fat},5} = 2, \quad d_{\text{Tall}\Diamond\text{Fat},6} = 1$$

and, moreover, one gets: $k_{\text{Tall}\Diamond\text{Fat}} = k_{\text{Tall}} + k_{\text{Fat}} = 1 + 1 = 2$.

In turn, as regards the second parts, one gets:

	[0.8,1,1]	[0.5,0.7,0.9]·	[0.2,0.4,0.6]	[0,0,0.2]	©	
		[0.8,1,1]	[0.2,0.4,0.6]	[0,0,0.2]	=	
	$a_{4,1}$	$a_{3,1}$	$a_{2,1}$	$a_{1,1}$		
	$a_{4,2}$	$a_{3,2}$	$a_{2,2}$	$a_{1,2}$		
$a_{4,3}$	$a_{3,3}$	$a_{2,3}$	$a_{1,3}$			
γ_6	γ_5	γ_4	γ_3	γ_2	γ_1	

where

$$a_{i,j} = d_{\text{Fat},j} * d_{\text{Tall},i-j+1} * (k_{\text{Tall}} * a_{i-j+1} + k_{\text{Fat}} * b_j)$$

For the second parts

$$\alpha_6\alpha_5\alpha_4\alpha_3\alpha_2\alpha_1 © \beta_3\beta_2\beta_1 = \gamma_8\gamma_7\gamma_6\gamma_5\gamma_4\gamma_3\gamma_2\gamma_1$$

$$a_{1,1} = d_{\mathrm{Tall}\Diamond\mathrm{Fat},1} * d_{\mathrm{AGILE},1} * (k_{\mathrm{Tall}\Diamond\mathrm{Fat}} * \alpha_1 + k_{\mathrm{AGILE}} * \beta_1)$$
$$= 1 * 1 * (2 * [0,0,0.2] + 1 * [0,0,0.2]) = [0,0,0.4] + [0,0,0.2] = [0,0,0.6],$$
$$a_{1,2} = [0.2,0.4,1], \quad a_{1,3} = [0.5,0.7,1.3], \quad a_{2,1} = [0.4,0.8,2],$$
$$a_{2,2} = [0.4,0.8,2], \quad a_{2,3} = [1.4,2.4,3.4],$$
$$a_{3,1} = [1.68,2.46,4.08], \quad a_{6,3} = [2.1,2.7,2.9],$$

$$\gamma_1 = [(k_{\mathrm{Tall}\Diamond\mathrm{Fat}} + k_{\mathrm{AGILE}}) * d_{\mathrm{Tall}\Diamond\mathrm{FatAgile},1}]^{-1} * a_{1,1} = [3 * 1]^{-1} * [0,0,0.6] = [0,0,0.2] \approx \mathrm{F},$$
$$\gamma_2 = (1/9) * ([0.2,0.4,1] + [0.4,0.8,2]) = [0.06,0.133,0.33] \approx \mathrm{ib[F,AT]},$$
$$\gamma_3 \approx \mathrm{ib[F,AT]}, \quad \gamma_4 \approx \mathrm{AT}, \quad \gamma_5 \approx \mathrm{ib[AT,T]}, \quad \gamma_6 \approx \mathrm{b[T]},$$
$$\gamma_7 \approx \mathrm{nt[T]}, \quad \gamma_8 \approx \mathrm{ib[T,CT]}.$$

The final result is

$$\mathrm{Tall}\Diamond\mathrm{Fat}\Diamond\mathrm{Agile} = \varnothing^{\mathrm{ib[T,CT]}}\varnothing^{\mathrm{nt[T]}}\{g\}^{\mathrm{b[T]}}\{d,f\}^{\mathrm{ib[AT,T]}} \times \{a,b,c,e\}^{\mathrm{AT}}\varnothing^{\mathrm{ib[F,AT]}}\varnothing^{\mathrm{ib[F,AT]}}\varnothing^{\mathrm{F}}(3,1,3,6,8,8,6,3,1).$$

It is apparent that there are similarities between our composition and the conventional arithmetical multiplications: our operation replaces the conventional multiplication of figures with set-theoretic intersection $\cap$, and the addition of numbers with the union of sets $\cup$. The way the composition operates, can be sketched as follows:

Multiplication of integers	Composition $A\Diamond B$		
	First parts	Additional factors	Second parts
Multiplication with carry	Intersection of sets $\cap$	No carry multiplication $d_{B,j} * d_{A,i+1-j}$	$d_{B,j} * d_{A,i-j+1}$ $*(k_A * \alpha_{i-j+1} + k_B * \beta_j)$
Addition with carry	Union of sets $\cup$	No carry addition	$1/((k_A + k_B) \cdot d_{c,i})$ $* \sum_{j=i-n+1,\dots,m}$

In the definition of the composition operator a vital role is played by the additional information present in each string. Suppose that the strings A and B are not initial, then:

- The indices dA_i and dB_i represent the number of sets whose union have generated the ith class of A and B, respectively.
- The indices kA and kB represent, in turn, the number of classes whose intersection have generated the classes of A and B, respectively.

Thus these indices include the history of our string.

$$1/((k_A + k_B) \cdot d_{c,i}) * \sum_{j=1,\ldots,i} dB_j * dA_{i-j+1} * (k_A * a_{i-j+1} + k_B * b_j)$$

represents essentially a mean among the labels [6] where each label takes a weight in some way related to its evolution. This approach allows to preserve the associativity for the operation $\Diamond$, this would not be possible if we considered just the average value of the labels.

The following properties hold:

Proposition 4.1.
(i) *The operation of composition $\Diamond$ is both commutative and associative and its unit element is U^{NC}.*
(ii) *The triple $(S(U), \Diamond, U^{\mathrm{NC}})$ is a commutative monoid.*

Proof. The proposition (i) is proved in [4,19], whereas (ii) stems directly from (i). □

5. The ordering relations

5.1. The relation on $P(U)$

Since our string consist of subsets of U, in order to define an ordering criterion among them, one has first to introduce an ordering criterion among sets.

Let us suppose that U is a totally ordered set, i.e., there is an ordering relation $\leqslant_U$ defined on U such that for every $x \in U$ and for every $y \in U$ one has either $x \leqslant_U y$ or $y \leqslant_U x$.

Let $X \subseteq U$, we define
$\max_1(X)$ the maximum of X as regards the relation $\leqslant_U$;
$\max_2(X)$ the second maximum of X;
$\max_3(X)$ the third maximum of X;
......
$\max_{|X|}(X)$ the minimum of X.
It is now possible to introduce the following:

Definition. Let X and Y be two subsets of U, then we say that $X \leqslant_S Y$ iff holds true either $X = Y$ or $|X| < |Y|$ or $|X| = |Y|$ and for some i, $1 \leqslant i \leqslant |X|$, one has:

$$\max_j(X) = \max_j(Y) \quad \text{for every } j < i \quad \text{and} \quad \max_i(X) \leqslant_U \max_i(Y).$$

Example. Consider $U = \{a, b, c, d, e, f, g\}$ and suppose $a <_U b <_U c <_U d <_U e <_U f <_U g$ then:

- if $X = \{a, d, g, e\}$ and $Y = \{a, b\}$, then $Y \leqslant_S X$ because $|Y| < |X|$;
- if $X = \{a, d, g, e\}$ and $Y = \{a, b, c, d\}$, then $Y \leqslant_S X$ because $\max_1(Y) = d <_U g = \max_1(X)$;
- if $X = \{a, d, g, e\}$ and $Y = \{a, b, c, g\}$, then $Y \leqslant_U X$ because $\max_2(Y) = c <_U g = \max_1(X)$.

Proposition 5.1. *The ordering relation* $\leqslant_S$ *on* $P(U)$ *is total.*

Proof. It is apparent that two subsets of U are always comparable with respect to $\leqslant_S$, moreover if $X \neq Y$ then either they have different cardinality or there is at least one element different and thus $\max_i(X) = \max_i(Y)$ cannot hold for every i. □

5.2. The ordering relation for attribute strings

In order to define the relation $\leqslant$ for the attribute strings, we note that the ordering is based on the information contained in each string and it can be evaluated according to the coefficients k in the strings. The more k is high, for a specific string, the more is high the number of strings whose composition has generated the string. Thus the more the content of information is high, the more the more is high the possibility for it to precede other strings in the ordering

Definition. Let $A = a_n^{\alpha_n} a_{n-1}^{\alpha_{n-1}} \ldots a_2^{\alpha_2} a_1^{\alpha_1}$ $(k_A, d_{A,n}, \ldots, d_{A,2}, d_{A,1})$ and $B = b_m^{\beta_m} b_{m-1}^{\beta_{m-1}} \ldots b_2^{\beta_2} b_1^{\beta_1}$ $(k_B, d_{B,m}, \ldots, d_{B,2}, d_{B,1})$ be two attribute string of the universe U, then $A \leqslant B$ if either $A = B$ or $k_A > k_B$ or $k_A = k_B$ and there is an index i $(1 \leqslant i \leqslant \min(n, m))$ such that for every $j < i$ $a_j = b_j$, $\alpha_j = \beta_i$ and $d_{A,j} = d_{B,j}$ whereas $a_i <_S b_i$ or $a_i = b_i$ $\alpha_i < \beta_i$ or $a_i = b_i$ and $\alpha_i = \beta_i$ and $d_{A,i} > d_{B,i}$ moreover for every $A \in S(U)$ we have $U^{\mathrm{NI}} \leqslant A \leqslant U^{\mathrm{NC}}$.

It is apparent that the strings U^{NC} and U^{NI} represent the minimum and the maximum, respectively. In general, this relation can be viewed as an ordering carried out with respect to the information contained in each string, and a string having a higher content of information than another one precedes the latter in the ordered sequence.

Example. If $A = \{a, b\}^{[0.8,0.9,1]}\{e, h, g\}^{[0.5,0.7,0.9]}\{c, d, f\}^{[0.2,0.4,0.6]}(2, 1, 2, 1)$ and $B = \{a, b, e, h, g\}^{[0.6,0.8,0.9]}\{c, d, f\}^{[0.3,0.6,0.8]}(1, 1, 1)$, then one has $k_A = 2 > 1 = k_B$ and thus $A \leqslant B$.

If $A = \{a,b\}^{[0.8,0.9,1]}\{e,h,g\}^{[0.5,0.7,0.9]}\{c,d,f\}^{[0.2,0.4,0.6]}(2,1,2,1)$ and $B = \{a,b,e,h\}^{[0.6,0.8,0.9]}\{c,d,f,g\}^{[0.3,0.6,0.8]}(2,1,1)$ then $A \leqslant B$ because $k_A = k_B$ and $\{c,d,f\} <_S \{c,d,f,g\}$.

If $A = \{a,b\}^{[0.8,0.9,1]}\{e,h,g\}^{[0.5,0.7,0.9]}\{c,d,f\}^{[0.2,0.4,0.6]}(2,1,2,1)$ and $B = \{a,b\}^{[0.7,0.75,0.8]}\{e,h,g\}^{[0.6,0.8,0.9]}\{c,d,f\}^{[0.3,0.6,0.8]}(2,1,1)$, then $A \leqslant B$ because $k_A = k_B, a_1 = \{c,d,f\} = \{c,d,f\} = b_1$ and $\alpha_1 = [0.2,0.4,0.6] < [0.3,0.6,0.8] = \beta_1$.

If $A = \{a,b\}^{[0.8,0.9,1]}\{e,h,g\}^{[0.5,0.7,0.9]}\{c,d,f\}^{[0.2,0.4,0.6]}(2,1,2,1)$ and $B = \{a,b\}^{[0.7,0.75,0.8]}\{e,h,g\}^{[0.5,0.7,0.9]}\{c,d,f\}^{[0.2,0.4,0.6]}(2,1,1)$ then $A \leqslant B$ because $k_A = k_B, a_1 = b_1, \alpha_1 = \beta 1, d_{A1} = 1 = 1 = d_{B1}, a_2 = b_2, \alpha_2 = \beta_2, d_{A2} = 2 > 1 = d_{B2}$.

Definition. A set is *complete* if every lower limited part is endowed with g.l.b. and every upper limited part is endowed with l.u.b. Consequently one has:

A set endowed with both minimum and maximum elements is complete if and only if every part is endowed with both g.l.b. and l.u.b.

The set $S(U)$ is endowed with both minimum and maximum elements, moreover the ordering of strings takes place by comparing the components of the strings themselves, namely
- subsets of U;
- fuzzy numbers in the interval $[0,1]$, in turn consisting of real numbers in $[0,1]$;
- integers (coefficients).

Since one has that:
- subsets of U are finite;
- the interval $[0,1]$ is complete;
- the infinite subsets of strings whose coefficients grow to infinite have as g.l.b. the string U^{NI};

one has the following proposition holds:

Proposition 5.2. *The structure* $(S(U), \Diamond, U^{\mathrm{NC}}, \leqslant)$ *is an integral commutative l-monoid.*

Proof. We prove (ii). $(S(U), \Diamond, U^{\mathrm{NC}}, \leqslant)$ is a *l-monoid* because in the monoid $(S(U), \Diamond, U^{\mathrm{NC}})$ a lattice $(S(U), \leqslant)$ with both minimum and maximum has been defined. Moreover, the lattice is commutative because the operation $\Diamond$ is commutative. It is also integral since the maximum U^{NC} of the lattice is at the same time the unit element of the monoid. □

6. The relevance

It is sometimes important to grasp an idea about the way a string attribute can affect the generation of a string, the concept of *relevance* is useful to this aim. When two strings A and B are composed into $A\Diamond B$ in order to get the string C essentially we "shake up" the information in A and B and put it into C. The relevance measures the ability of A in influencing B (and vice-versa) in order to get C. It is worth introducing two measures of relevance, as the operation of composition affects both first and second parts.

6.1. The relevance for first parts

The following definition was introduced by [21].

Let $A = a_n^{\alpha_n}a_{n-1}^{\alpha_{n-1}}\dots a_2^{\alpha_2}a_1^{\alpha_1}$ be an attribute string, and let C be the result of the composition of A with other attributes (e.g., $C = A\Diamond B\Diamond D$), then the relevance μ_A^C of A with respect to C is defined as follows:

$$\mu_A^C = \sum_{j=1,\dots,n}\sum_{k=1,\dots,m}(p_{jk} - p_j)^2,$$

where $p_{jk} = \#[A^{-1}(\alpha_j)\cap c_k]$ and $p_j = \#[A^{-1}(\alpha_j)]/m$ and c_k belongs to the set of the first points of the string C.

From the conceptual point of view the relevance of the attribute A represents the quantity of information conveyed by A into the attribute C or, equivalently, the ability of A in singling out the elements for the generation of the attribute C. The formula at first glance appears similar to the one for the variance, in fact we compute the extent to which the distribution of the classes of A in C differs from that in which the distribution of A is uniform.

Example (*see Appendix* B). Let

$$A = \{c,g\}^{CT}\{d\}^{T}\{b,e,f\}^{AT}\{a\}^{F}(1,1,1,1,1),$$
$$B = \{g\}^{CT}\{a,d,e,f\}^{AT}AT\{b,c\}^{F}(1,1,1,1),$$
$$C = A\Diamond B$$
$$= \{g\}^{CT}\varnothing^{AT}\{c,d\}^{ib[AT,T]}\{e,f\}^{AT}\{a,b\}^{ib[F,AT]}\varnothing^{F}(2,1,2,3,3,2,1).$$

We compute $\mu_A^C = \sum_{j=1,\dots,4}\sum_{k=1,\dots,6}(p_{jk} - p_j)^2$, one has: $p_1 = \#[\{a\}]/6 = 1/6$, $p_2 = \#[\{b,e,f\}]/6 = 3/6 = 1/2, p_3 = 1/6, p_4 = 1/3$

$$\begin{array}{llll} p_{1,1} = \#[\{a\} \cap \emptyset] = 0 & p_{2,1} = 0 & p_{3,1} = 0 & p_{4,1} = 0 \\ p_{1,2} = \#[\{a\} \cap \{a,b\}] = 1 & p_{2,2} = 1 & p_{3,2} = 0 & p_{4,2} = 0 \\ p_{1,3} = \#[\{a\} \cap \{e,f\}] = 0 & p_{2,3} = 2 & p_{3,3} = 0 & p_{4,3} = 0 \\ p_{1,4} = 0 & p_{2,4} = 0 & p_{3,4} = 1 & p_{4,4} = 1 \\ p_{1,5} = 0 & p_{2,5} = 0 & p_{3,5} = 0 & p_{4,5} = 0 \\ p_{1,6} = 0 & p_{2,6} = 0 & p_{3,6} = 0 & p_{4,6} = 1 \end{array}$$

thus

$$\begin{aligned} \mu_A^C &= (0-1/6)^2 + (1-1/6)^2 + (0-1/6)^2 + (0-1/6)^2 + (0-1/6)^2 \\ &\quad + (0-1/6)^2 + (0-1/2)^2 + (1-1/2)^2 + (2-1/2)^2 + (0-1/2)^2 \\ &\quad + (0-1/2)^2 + (0-1/2)^2 + (0-1/6)^2 + (0-1/6)^2 + (0-1/6)^2 \\ &\quad + (1-1/6)^2 + (0-1/6)^2 + (0-1/6)^2 + (0-1/3)^2 + (0-1/3)^2 \\ &\quad + (0-1/3)^2 + (1-1/3)^2 + (0-1/3)^2 + (1-1/3)^2 \\ &= 1/36 + 25/36 + 1/36 + 1/36 + 1/36 + 1/36 + 1/4 + 1/4 + 9/4 \\ &\quad + 1/4 + 1/4 + 1/4 + 1/36 + 1/36 + 1/36 + 25/36 + +1/36 + 1/36 \\ &\quad + 1/9 + 1/9 + 1/9 + 4/9 + 1/9 + 4/9 \\ &= 5/6 + 7/2 + 5/6 + 4/3 = 6.5. \end{aligned}$$

In a similar way (see Appendix C) one gets $\mu_B^C = 5.5$.

The conclusion is that A affects the composition of C more than B.

6.2. The relevance for the second parts

We note that the relevance for the first parts does not take into account the labels, since it operates just on partitions. However, it is possible to evaluate how labels affect the operation $\Diamond$. We remember that the operation for second parts is based on the mean among fuzzy numbers, and in case a string consists of many clusters their labels will appear often in the computation of means and the more the labels will be important the more they are nearby.

Let us consider how the computation of labels is carried out for the composition $\Diamond$:

$$\gamma_i = 1/((k_A + k_B) \cdot d_{c,i}) * \sum_j d_{B,j} * d_{A,i-j+1} * (k_A * \alpha_{i-j+1} + k_B * \beta_j).$$

The more the label α_i of the string A is relevant the more the coefficients k_A and d_{Ai} are high. Thus the relevance of a string is to be directly proportional to $k_A + \sum d_{Ai}$, and inversely proportional to the dispersion of the label on the interval $[0, 1]$, representable as follows:

$$\frac{\sum_{i=1,\dots,n-1}\sum_{j=i+1,\dots,n}|\alpha_i-\alpha_j|}{n*(n-1)/2},$$

where if $\alpha_i = [\alpha_{i1}, \alpha_{i2}, \alpha_{i3}]$ and $\alpha_j = [\alpha_{j1}, \alpha_{j2}, \alpha_{j3}]$, then $|\alpha_i - \alpha_j| = |\alpha_{i1} - \alpha_{j1}| + |\alpha_{i2} - \alpha_{j2}| + |\alpha_{i3} - \alpha_{j3}|$.

The sum is divided by $n*(n-1)/2$ to find the mean of all these differences. Given a string $A = a_n^{\alpha_n} a_{n-1}^{\alpha_{n-1}} \dots a_2^{\alpha_2} a_1^{\alpha_1}$ we can define the relevance for the second parts as follows:

$$\rho_A = \begin{cases} 0 & \text{if } A = U^{\mathrm{NC}}, \\ 1 & \text{if } A = U^{\mathrm{NI}}, \\ 2/\pi \operatorname{arctg} \frac{n*(n-1)\left(k_A + \sum_{i=1,\dots,n} d_{Ai}\right)}{2*\sum_{i=1,\dots,n-1}\sum_{j=i+1,\dots,n}|\alpha_i-\alpha_j|} & \text{otherwise.} \end{cases}$$

The strings U^{NC} and U^{NI} attain, respectively, the minimum and maximum value of the relevance. The relevance of U^{NC} equals zero because this string is does not affect the string to which it is applied ($U^{\mathrm{NC}} \Diamond A = A$ for every A), whereas the relevance for U^{NI} equals 1 because "absorbs" any string ($U^{\mathrm{NI}} \Diamond A = U^{\mathrm{NI}}$ for every A).

It is worth emphasizing that the relevance ρ_A is computed beginning from only the information present in A. In other words, the relevance μ_A^C is the relevance of A with respect to a resulting string C, whereas ρ_A is the relevance of A in absolute terms. Thus μ if the *relative relevance* and ρ is *the absolute relevance* (Appendix D contains an exhaustive example for the computation of the relevance).

6.3. The total relevance

Beginning from the relevance μ and the relevance ρ one can get a new measure, combination of μ and ρ. The *total relevance* of A with respect to C is

$$R_A^C = \mu_A^C * \rho_A.$$

Example. In the previous example we have computed $\mu_A^C = 6.5$ and $\mu_B^C = 5.5$. We note that F = [0, 0, 0.2], AT = [0.2, 0.4, 0.6], T = [0.5, 0.7, 0.9], CT = [0.8, 1, 1], thus we get for ρ_A:

$$\begin{aligned}&(|0-0.2|+|0-0.4|+|0.2-0.6|)+(|0-0.5|+|0-0.7|+|0.2-0.9|)\\ &+(|0-0.8|+|0-1|+|0.2-1|)+(|0.2-0.5|+|0.4-0.7|\\ &+|0.6-0.9|)+(|0.2-0.8|+|0.4-1|+|0.6-1|)+(|0.5-0.8|\end{aligned}$$

Table 1
Constitution of 38 younsters 13, 14 and 15 years old[a]

Boy	13 years				14 years				15 years			
	T (cm)	*W* (kg)	*C* (cm)	*S* (cm)	*T* (cm)	*W* (kg)	*C* (cm)	*S* (cm)	*T* (cm)	*W* (kg)	*C* (cm)	*S* (cm)
1	147	40	70	80	157	47	76	85	162	54	81	87
2	162	49	75	85	166	50	75	87	167	52	79	88
3	153	45	72	86	159	48	75	90	161	51	75	92
4	155	51	77	85	163	58	82	87	168	66	87	92
5	160	51	75	86	165	56	77	88	167	61	82	89
6	153	38	67	81	159	43	70	84	167	44	71	87
7	166	67	86	89	169	72	89	90	172	79	92	95
8	168	55	76	91	174	60	79	93	175	65	81	95
9	142	35	69	75	149	39	68	78	155	46	75	82
10	151	44	72	79	160	51	78	85	165	57	80	89
11	164	55	77	88	167	58	79	89	169	65	80	93
12	153	42	70	83	163	46	73	88	168	53	78	91
13	148	41	72	78	158	47	77	82	164	51	81	85
14	164	75	92	90	169	84	97	93	171	88	102	95
15	145	34	65	76	151	39	68	80	162	45	72	84
16	151	51	80	81	159	57	83	85	162	64	87	87
17	145	50	82	79	153	55	84	81	162	59	82	86
18	154	47	71	82	163	53	75	86	169	56	80	89
19	156	48	73	81	166	50	72	86	171	56	75	89
20	144	30	60	73	149	33	62	75	157	37	66	79
21	154	41	69	82	164	49	76	88	169	56	77	91
22	155	43	71	82	165	52	75	87	169	57	79	90
23	155	48	76	82	162	58	85	86	166	60	84	89
24	155	49	73	80	162	55	76	84	172	57	76	87
25	156	48	76	81	163	53	79	86	164	54	82	87
26	156	50	74	81	164	53	76	84	172	56	79	87
27	162	45	71	84	168	48	71	88	170	52	75	89
28	147	37	71	79	154	43	75	82	163	50	80	86

(Continued on next page)

Table 1 (*continued*)

Boy	13 years				14 years				15 years			
	T (cm)	W (kg)	C (cm)	S (cm)	T (cm)	W (kg)	C (cm)	S (cm)	T (cm)	W (kg)	C (cm)	S (cm)
29	149	40	71	80	157	47	79	83	166	53	78	87
30	148	37	69	78	155	41	70	81	162	47	74	85
31	156	52	75	83	163	57	79	87	166	62	81	89
32	141	35	68	73	151	42	74	77	159	48	79	82
33	140	30	67	76	147	34	70	77	157	43	73	83
34	146	149	76	80	153	52	78	79	161	53	76	84
35	162	53	74	86	168	58	78	79	161	53	76	84
36	146	36	68	77	158	44	75	85	165	51	73	89
37	141	41	71	76	151	46	75	80	158	51	76	83
38	158	65	93	85	167	71	93	90	171	79	90	91

[a] T = Tallness; W = weight; C = chest; S = Tallnessseated.

Table 2
Three-way data fuzzy clustering

Boy	Degree of each cluster				Boy	Degree of each cluster			
	A	*B*	*C*	*D*		*A*	*B*	*C*	*D*
1	0.01	0.08	0.81	0.10	20	0.03	0.71	0.16	0.10
2	0.03	0.03	0.14	0.81	21	0.03	0.05	0.28	0.64
3	0.05	0.10	0.38	0.47	22	0.01	0.03	0.19	0.77
4	0.13	0.06	0.22	0.60	23	0.06	0.05	0.28	0.61
5	0.04	0.03	0.12	0.80	24	0.03	0.05	0.30	0.61
6	0.03	0.22	0.49	0.26	25	0.02	0.04	0.34	0.59
7	0.95	0.01	0.02	0.03	26	0.03	0.04	0.22	0.71
8	0.34	0.08	0.18	0.40	27	0.05	0.07	0.22	0.67
9	0.01	0.94	0.04	0.02	28	0.22	0.27	0.61	0.11
10	0.01	0.03	0.78	0.18	29	0.01	0.04	0.89	0.07
11	0.18	0.06	0.18	0.58	30	0.01	0.63	0.27	0.09
12	0.03	0.06	0.33	0.59	31	0.03	0.03	0.15	0.79
13	0.01	0.10	0.79	0.10	32	0.01	0.83	0.11	0.05
14	0.86	0.03	0.05	0.07	33	0.01	0.89	0.07	0.03
15	0.01	0.87	0.08	0.04	34	0.04	0.24	0.55	0.18
16	0.10	0.10	0.42	0.39	35	0.09	0.13	0.33	0.45
17	0.07	0.17	0.49	0.26	36	0.02	0.23	0.58	0.17
18	0.01	0.02	0.17	0.80	37	0.02	0.68	0.23	0.07
19	0.03	0.05	0.22	0.70	38	0.87	0.02	0.04	0.06

7.2. The attribute strings

Let us define the attribute strings which represent the classifications of the universe. More specifically, let $T13$, $W13$, $C13$ and $S13$ denote, respectively, the strings related to tallness, weight, chest, and tallness when seated concerning boys who are 13. The same notation is adopted for boys who are 14 and 15.

$$T13 = \{7,8\}^{\mathrm{vt}}\{11,14\}^{\mathrm{ib[t,vt]}}\{27,35,2,5\}^{\mathrm{alto}}\{38\}^{\mathrm{ib[a1,t]}} \times \{19,22,23,24,25,26,31\}^{\mathrm{nt[a1]}}\{3,4,6,10,12,16,18,21\}^{\mathrm{a1}} \times \{1,28,13,29,30\}^{\mathrm{b[a1]}}\{36\}^{\mathrm{ib[s,a1]}}\{15,17,20,34\}^{\mathrm{nt[s]}}\{33,32,37,9\}^{\mathrm{s}},$$

$$W13 = \{14\}^{\mathrm{vh}}\{7,38\}^{\mathrm{ib[h,vh]}}\{8,11\}^{\mathrm{h}}\{31,35\}^{\mathrm{ib[a2,h]}}\{4,5,16,17,26\}^{\mathrm{nt[a2]}} \times \{2,18,19,23,24,25,34\}^{a2}\{3,10,12,13,21,22,27,37\}^{\mathrm{b[a2]}} \times \{1,6,28,29,30,36\}^{\mathrm{ib[l,a2]}}\{9,15,32\}^{\mathrm{nt[l]}}\{20,33\}^{\mathrm{l}},$$

$$C13 = \{14,38\}^{\mathrm{vb}}\{7\}^{\mathrm{ib[b,vb]}}\{16,17\}^{\mathrm{b}}\{2,4,5,8,11,23,25,31,34\}^{\mathrm{nt[a3]}} \times \{1,3,10,12,13,18,19,22,24,26,27,28,29,35,37\}^{\mathrm{a3}} \times \{6,9,15,21,30,32,36\}^{\mathrm{ib[s,a3]}}\{33\}^{\mathrm{nt[s]}}\{20\}^{\mathrm{s}},$$

$$S13 = \{8\}^{\mathrm{vt}^\circ}\{7,11,14\}^{\mathrm{b}[\mathrm{vt}^\circ]}\{27\}^{\mathrm{nt}[\mathrm{t}^\circ]}\{2,3,4,5,35,38\}^{\mathrm{t}^\circ} \times \{12,18,21,22,23,31\}^{\mathrm{ib}[\mathrm{a}^\circ,\mathrm{t}^\circ]} \times \{1,6,10,16,17,19,24,25,26,28,29,34\}^{\mathrm{a}^\circ}\{13,30\}^{\mathrm{b}[\mathrm{a}^\circ]}\{36\}^{\mathrm{ib}[\mathrm{s}^\circ,\mathrm{a}^\circ]} \times \{9,20,32\}^{\mathrm{s}^\circ},$$

$$T14 = \{8\}^{\mathrm{vt}}\{7,11,14,27,35,38\}^{\mathrm{t}}\{2,19\}^{\mathrm{b}[\mathrm{t}]} \times \{4,12,18,21,23,24,25,26,31\}^{\mathrm{a}1}\{1,3,6,10,13,16,29,36\}^{\mathrm{b}[\mathrm{a}1]} \times \{28,30\}^{\mathrm{ib}[\mathrm{s},\mathrm{a}1]}\{115,17,32,34,37\}^{\mathrm{nt}[\mathrm{s}]}\{9,20,33\}^{\mathrm{s}},$$

$$W14 = \{14\}^{\mathrm{vh}}\{7,38\}^{\mathrm{ib}[\mathrm{h},\mathrm{vh}]}\{4,8,11,23,35\}^{\mathrm{b}[\mathrm{h}]}\{5,16,17,24,31\}^{\mathrm{ib}[\mathrm{a}2,\mathrm{h}]} \times \{18,22,25,26,34\}^{\mathrm{nt}[\mathrm{a}2]}\{1,2,3,10,13,19,21,27,29\}^{\mathrm{a}2}\{12,37\}^{\mathrm{b}[\mathrm{a}2]} \times \{6,28,32,36\}^{\mathrm{ib}[\mathrm{l},\mathrm{a}2]}\{9,15,30\}^{\mathrm{nt}[\mathrm{l},\mathrm{a}2]}\{20,33\}^{\mathrm{l}},$$

$$C14 = \{14\}^{\mathrm{vb}}\{38\}^{\mathrm{b}[\mathrm{vb}]}\{7\}^{\mathrm{ib}[\mathrm{b},\mathrm{vb}]}\{8,10,11,25,29,31,34,35\}^{\mathrm{b}[\mathrm{b}]}\{5,13\}^{\mathrm{ib}[\mathrm{a}3,\mathrm{b}]} \times \{1,2,3,18,21,22,24,26,28,32,36,37\}^{\mathrm{a}3}\{12,19,27\}^{\mathrm{b}[\mathrm{a}3]} \times \{6,9,15,30,33\}^{\mathrm{ib}[\mathrm{s},\mathrm{a}3]}\{20\}^{\mathrm{s}},$$

$$S14 = \{8,14\}^{\mathrm{vt}^\circ}\{3,7,38\}^{\mathrm{ib}[\mathrm{t}^\circ,\mathrm{vt}^\circ]}\{2,4,5,11,12,21,22,27,31\}^{\mathrm{t}^\circ} \times \{18,19,23,25\}^{\mathrm{nt}[\mathrm{a}^\circ]}\{1,6,10,16,24,26,36\}^{\mathrm{a}^\circ} \times \{13,17,28,29,30\}^{\mathrm{b}[\mathrm{a}^\circ]}\{15,34,35,37\}^{\mathrm{ib}[\mathrm{s}^\circ,\mathrm{a}^\circ]}\{9,32,33\}^{\mathrm{nt}[\mathrm{s}^\circ]}\{20\}^{\mathrm{s}^\circ},$$

$$T15 = \{8\}^{\mathrm{vh}}\{7,14,19,24,26\}^{\mathrm{ib}[\mathrm{h},\mathrm{vh}]}\{27,38\}^{\mathrm{nt}[\mathrm{h}]}\{11,18,21,22\}^{\mathrm{h}} \times \{2,4,5,6,12\}^{\mathrm{b}[\mathrm{h}]}\{10,13,23,25,29,31,36\}^{\mathrm{a}1} \times \{1,3,15,16,17,28,30,34,35\}^{\mathrm{b}[\mathrm{a}1]}\{32,37\}^{\mathrm{nt}[\mathrm{s}]}\{9,20,33\}^{\mathrm{s}},$$

$$W15 = \{14\}^{\mathrm{vh}}\{7,38\}^{\mathrm{ib}[\mathrm{h},\mathrm{vh}]}\{4,8,11\}^{\mathrm{h}}\{16,31\}^{\mathrm{b}[\mathrm{h}]}\{5,17,23\}^{\mathrm{ib}[\mathrm{a}2,\mathrm{h}]} \times \{10,18,19,21,22,24,26\}^{\mathrm{nt}[\mathrm{a}2]}\{1,25\}^{a2} \times \{2,3,12,13,27,28,29,34,35,36,37\}^{\mathrm{b}[\mathrm{a}2]}\{6,9,15,30,32,33\}^{\mathrm{ib}[\mathrm{l},\mathrm{a}2]} \times \{20\}^{\mathrm{l}},$$

$$C15 = \{14\}^{\mathrm{vb}}\{7,38\}^{\mathrm{ib}[\mathrm{b},\mathrm{vb}]}\{4,16\}^{\mathrm{b}}\{23\}^{\mathrm{ib}[\mathrm{a}3,\mathrm{b}]}\{5,17,25\}^{\mathrm{nt}[\mathrm{a}3]} \times \{1,2,8,10,11,13,18,22,28,31,32\}^{\mathrm{a}3}\{12,21,26,29\}^{\mathrm{b}[\mathrm{a}3]} \times \{3,9,19,24,27,30,34,35,37\}^{\mathrm{ib}[\mathrm{s},\mathrm{a}3]}\{6,15,33,36\}^{\mathrm{s}}\{20\}^{\mathrm{s}},$$

$$
\begin{aligned}
S15 = {} & \{7,8,14\}^{\mathrm{vt}^\circ}\{3,4,11\}^{\mathrm{ib}[\mathrm{t}^\circ,\mathrm{vt}^\circ]}\{12,21,38\}^{\mathrm{nt}[\mathrm{t}^\circ]} \\
& \times \{5,10,18,19,22,23,27,31,36\}^{\mathrm{t}^\circ}\{2\}^{\mathrm{ib}[\mathrm{a}^\circ,\mathrm{t}^\circ]} \\
& \times \{1,6,16,17,24,25,26,29\}^{\mathrm{a}^\circ}\{13,15,28,30,34,35\}^{\mathrm{b}[\mathrm{a}^\circ]} \\
& \times \{9,32,33,37\}^{\mathrm{ib}[\mathrm{s}^\circ,\mathrm{a}^\circ]}\{20\}^{\mathrm{s}^\circ}.
\end{aligned}
$$

7.3. Composing the strings

Beginning from data present in the strings, we aim at obtaining classifications related to the constitution of boys for each age. We carry out the following compositions:

$$
\begin{aligned}
& C13 = T13\Diamond W13\Diamond C13\Diamond S13, \quad C14 = T14\Diamond W14\Diamond C14\Diamond S14, \\
& C15 = T15\Diamond W15\Diamond C15\Diamond S15,
\end{aligned}
$$

where $C13$, $C14$, $C15$ are the attribute strings related to the constitution at 13, 14 and 15. The computation furnishes the following results:

$$
\begin{aligned}
C13 = {} & \{14\}^{\mathrm{b[st]}}\{7,8\}^{\mathrm{ib[rs,st]}}\{11,38\}^{\mathrm{nt[rs]}}\{5,35\}^{\mathrm{b[rs]}}\{2,4,16,23,27,31\}^{\mathrm{ib[a4,rs]}} \\
& \times \{3,18,19,22,24,25,26\}^{\mathrm{nt[a4]}}\{10,12,17,21\}^{\mathrm{a4}} \\
& \times \{1,6,13,28,29,34\}^{\mathrm{b[a4]}}\{15,30,36,37\}^{\mathrm{ib[w,a4]}}\{9,23,33\}^{\mathrm{nt[w]}}\{20\}^{\mathrm{w}},
\end{aligned}
$$

$$
\begin{aligned}
C14 = {} & \{14\}^{\mathrm{b[st]}}\{38\}^{\mathrm{ib[rs,st]}}\{7,8\}^{\mathrm{nt[rs]}}\{4,11\}^{\mathrm{b[rs]}} \\
& \times \{2,5,16,19,22,23,25,27,31,35\}^{\mathrm{ib[a4,rs]}} \\
& \times \{3,10,17,18,21,24,26\}^{\mathrm{nt[a4]}}\{1,12,13,29\}^{\mathrm{a4}}\{34,36\}^{\mathrm{b[a4]}} \\
& \times \{6,15,28,30,32,37\}^{\mathrm{ib[w,a4]}}\{9,33\}^{\mathrm{nt[w]}}\{20\}^{\mathrm{w}},
\end{aligned}
$$

$$
\begin{aligned}
C15 = {} & \{14\}^{\mathrm{b[st]}}\{7\}^{\mathrm{ib[rs,st]}}\{4,8,38\}^{\mathrm{nt[rs]}}\{11\}^{\mathrm{rs}} \\
& \times \{5,16,18,19,21,22,23,31\}^{\mathrm{ib[a4,rs]}}\{10,12,17,24,26,27\}^{\mathrm{nt[a4]}} \\
& \times \{1,2,3,25,36\}^{\mathrm{a4}}\{6,13,28,29\}^{\mathrm{b[a4]}}\{9,15,30,32,34,35,37\}^{\mathrm{ib[w,a4]}} \\
& \times \{33\}^{\mathrm{nt[w]}}\{20\}^{\mathrm{w}}.
\end{aligned}
$$

It is worth noting that empty clusters are not present in the strings. Thus to get the final result we compute $C = C13\Diamond C14\Diamond C15$ and we obtain

$$C = \{14\}^{b[st]}\{7\}^{ib[rs,st]}\{8,38\}^{nt[rs]}\{4,11\}^{rs} \times \{2,5,16,18,19,22,23,27,31\}^{ib[a4,rs]}\{3,10,17,21,24,25,26,35\}^{nt[a4]} \times \{12\}^{a4}\{1,13,29,36\}^{b[a4]}\{6,9,15,28,30,32,34,37\}^{ib[w,a4]}\{33\}^{nt[w]} \times \{20\}^{w}.$$

7.4. Comparing the results

We wish to compare our results with those reported in [28] and summarized in Table 2. Our results are

$$C = \{14\}^{b[st]}\{7\}^{ib[rs,st]}\{8,38\}^{nt[rs]}\{4,11\}^{rs} \times \{2,5,16,18,19,22,23,27,31\}^{ib[a4,rs]}\{3,10,17,21,24,25,26,35\}^{nt[a4]} \times \{12\}^{a4}\{1,13,29,36\}^{b[a4]}\{6,9,15,28,30,32,34,37\}^{ib[w,a4]} \times \{33\}^{nt[w]}\{20\}^{w}.$$

We note that for almost all individuals the results are identical. If we attach the labels *strong*, *rather strong*, *average*, *weak* to clusters A, D, C, B, respectively, the differences in the results are very small. Attaching these labels to the clusters obtained via the 3-*way data fuzzy clustering* is not arbitrary. In fact, we remember that cluster A includes boys with strong constitution, C and D average and B weak. However, our clustering is more "readable", for example, it is apparent that the individual 33 is slightly stronger that 20 and this result is coherent with Table 2, whereas 20 is present in cluster B with a membership degree greater than 33.

In conclusion we can adfirm that our clustering methodology is satisfactory since grades adequately all individuals and attains the same levels of fuzzy clustering with additional advantage that results are immediately understandable thanks to linguistic labels.

7.5. The relevance

Given the attribute strings $C13$, $C14$ and $C15$ one could be interested in singling out the one which has given the most *relevant* contribution to the final result C. In other terms, we wish to investigate how the constitutions of the boys at the different ages have affected the global assessment.

Thus we have to compute $R^C_{C13}, R^C_{C14}, R^C_{C15}$. We have

$$\mu^C_{C13} = 62.32, \quad \mu^C_{C14} = 66.90, \quad \mu^C_{C15} = 44.41, \quad \rho_{C13} = 0.947166,$$
$$\rho_{C14} = 0.947166, \quad \rho_{C15} = 0.946477.$$

The final result is

$$R^C_{C13} = \mu^C_{C13} * \rho_{C13} = 62.32 * 0.947166 = 59.02738,$$
$$R^C_{C14} = \mu^C_{C14} * \rho_{C14} = 66.90 * 0.947166 = 63.3654,$$
$$R^C_{C15} = \mu^C_{C15} * \rho_{C15} = 44.41 * 0.946477 = 42.033.$$

We see that the most relevant string is $C14$. However, the difference between $C13$ and $C14$ is slight whereas the difference with $C15$ is high. This fact implies that the final constitution is mostly affected by their constitution at 13 and 14. It is worth emphasizing that our measure of relevance is unable to evaluate the attributes, but measures how much the classification induced by the attribute is affected by the latter.

8. Resolving the equation $A \Diamond X = B$

In this section we are going to introduce a new operation in order to solve equations in $S(U)$. In some way this new operation reminds us the division for real numbers, ust as the operation $\Diamond$ resembles to the multiplication. This approach allows to state some existence theorems for the composition of strings in $S(U)$.

8.1. Motivation

Suppose that we are evaluating a sample of individuals with respect to a specific pathology **c** and in connection with the pathologies **a** and **b**. Suppose that the incidence of **c** and **a** on our sample is known and it is represented by the strings C and A, respectively. We aim at determining how the pathology **b** affects the string C, i.e., we look for the string B such that $A \Diamond B = C$.

It is worth emphasizing that in such way one could, given a "goal" classification C, one can get it beginning from a classification A. Thus one can solve equations in $(S(U), \Diamond)$, where the attribute string plays the role of unknown variable. However, given two strings A and C, we are not sure that there is a string B such that $A \Diamond B = C$. In this section we are going to give a formal approach to the problem taking into account first the second parts and then the first parts.

8.2. The operation % for the second parts

The operation for the second parts will be denoted by the symbol "$\circ$" and its result, if it exists, is got by solving a system of equations. Consider the following:

Definition. Let

$$\alpha_n\alpha_{n-1}\ldots\alpha_1\ (k_A, d_{A1}, d_{A2}, \ldots, d_{An}) \quad \text{and} \quad \gamma_s\gamma_{s-1}\ldots\gamma_1\ (k_C, d_{C1}, d_{C2}, \ldots, d_{Cs})$$

be, respectively, the second parts of two string A and C such that $s-n+1>0$, then we define the binary operation % as follows:

$$\begin{aligned}&[\alpha_n\alpha_{n-1}\ldots\alpha_1\ (k_A, d_{A1}, d_{A2}, \ldots, d_{An})]\%[\gamma_s\gamma_{s-1}\ldots\gamma_1\ (k_C, d_{C1}, d_{C2}, \ldots, d_{Cs})]\\ &\quad = [\beta_{s-n+1}\beta_{s-n}\ldots\beta_1\ (k_B, d_{B1}, d_{B2}, \ldots, d_{Bs-n+1})],\end{aligned}$$

where $k_B = k_C - k_A$ and the values d_{ci} are either the possible solution of the following system of s equations in $s-n+1$ unknowns

$$d_{Ci} = \begin{cases} \sum_{j=1,\ldots,i}(d_{Ai+1-j} * d_{Bj}), & 1 \leqslant i \leqslant s-n,\\ \sum_{i=1,\ldots,s-n+1}(d_{Ai+1-j} * d_{Bj}), & s-n+1 \leqslant i \leqslant n-1,\\ \sum_{j=i-n+1,\ldots,s-n+1}(d_{Ai+1-j} * d_{Bj}), & n \leqslant i \leqslant s \end{cases}$$

if $n > s-n+1$ or of the system

$$d_{Ci} = \begin{cases} \sum_{j=1,\ldots,i}(d_{Ai+1-j} * d_{Bj}), & 1 \leqslant i \leqslant n-1,\\ \sum_{j=i-n+1,\ldots,i}(d_{Ai+1-j} * d_{Bj}), & n \leqslant i \leqslant s-n,\\ \sum_{j=i-n+1,\ldots,s-n+1}(d_{Ai+1-j} * d_{Bj}), & s-n+1 \leqslant i \leqslant s \end{cases}$$

if $n \leqslant s-n+1$.

In turn, the values β_i are computed either solving the following system:

$$\gamma_i = \begin{cases} 1/((k_A+k_B)\cdot d_{c,i}) * \sum_{j=1,\ldots,i} d_{Bj} * d_{Ai-j+1} * (k_A * a_{i-j+1} + k_B * b_j),\\ \quad 1 \leqslant i \leqslant s-n,\\ 1/((k_A+k_B)\cdot d_{c,i}) * \sum_{j=1,\ldots,s-n+1} d_{Bj} * d_{Ai-j+1} * (k_A * a_{i-j+1} + k_B * b_j),\\ \quad s-n+1 \leqslant i \leqslant n-1,\\ 1/((k_A+k_B)\cdot d_{c,i}) * \sum_{j=i-n+1,\ldots,s-n+1} d_{Bj} * d_{Ai-j+1} * (k_A * a_{i-j+1} + k_B * b_j),\\ \quad n \leqslant i \leqslant s \end{cases}$$

if $n > s-n+1$ or the system:

$$\gamma_i = \begin{cases} 1/((k_A+k_B)\cdot d_{c,i}) * \sum_{j=1,\ldots,i} d_{Bj} * d_{Ai-j+1} * (k_A * a_{i-j+1} + k_B * b_j),\\ \quad 1 \leqslant i \leqslant n-1\\ 1/((k_A+k_B)\cdot d_{c,i}) * \sum_{j=i-n+1,\ldots,i} d_{Bj} * d_{Ai-j+1} * (k_A * a_{i-j+1} + k_B * b_j),\\ \quad n \leqslant i \leqslant s-n\\ 1/((k_A+k_B)\cdot d_{c,i}) * \sum_{j=i-n+1,\ldots,s-n+1} d_{Bj} * d_{Ai-j+1} * (k_A * a_{i-j+1} + k_B * b_j),\\ \quad s-n+1 \leqslant i \leqslant s \end{cases}$$

if $n \leqslant s-n+1$.

This new operation is defined if and only if the previous systems have solution and the following propositions hold true:
1. the quantities β_i are triangular numbers included in the interval $[0,1]$;
2. $k_B > 0$ and $d_{Bi} > 0$ for every i, moreover the values k_B and d_{Bi} are integers.

Example 1. Consider $\alpha_2\alpha_1\ (k_A, d_{A2}, d_{A1}) = [0.5, 0.7, 0.9][0.3, 0.5, 0.8](3, 2, 2)$ and let $\gamma_3\gamma_2\gamma_1\ (k_C, d_{C3}, d_{C2}, d_{C1}) = [0.4, 0.6, 0.7][0.3, 0.5, 0.6][0.2, 0.4, 0.5](5, 3, 4, 3)$.

We have $s - n + 1 = 3 - 2 + 1 = 2$ and consequently $n = 3 > s - n + 1$. We solve the system

$$d_{C1} = d_{A1} * d_{B1}, \quad d_{C2} = d_{A1} * d_{B2} + d_{A2} * d_{B1}, \quad d_{C3} = d_{A2} * d_{B2}.$$

Replacing the values one has

$$3 = 2 * d_{B1}, \quad 4 = 2 * d_{B2} + 2 * d_{B1}, \quad 3 = 2 * d_{B2}.$$

Because it is required that $d_{B1} = 3/2$ and $d_{B2} = 3/2$ we can adfirm that $\alpha_2\alpha_1(k_A, d_{A2}, d_{A1})\%\gamma_3\gamma_2\gamma_1(k_C, d_{C3}, d_{C2}, d_{C1})$ is not defined.

Example 2. Suppose that $\alpha_4\alpha_3\alpha_2\alpha_1(k_A, d_{A4}, d_{A3}, d_{A2}, d_{A1}) = [0.8, 1, 1][0.5, 0.7, 0.9]$ $[0.2, 0.4, 0.6][0, 0, 0.2](1, 1, 1, 1, 1)$ and $\gamma_6\gamma_5\gamma_4\gamma_3\gamma_2\gamma_1(k_C, d_{C6}, d_{C5}, d_{C4}, d_{C3}, d_{C2}, d_{C1}) =$ $[0.8, 1, 1][0.5, 0.8, 0.9][0.4, 0.6, 0.7][0.2, 0.4, 0.6][0.1, 0.2, 0.4][0, 0, 0.2](2, 1, 2, 3, 3, 2, 1)$ $s - n + 1 = 6 - 4 + 1 = 3 > 0$. Moreover $n > 4 > s - n + 1$.

We compute the quantities d_{Bi}

$$d_{C1} = d_{A1} * d_{B1}, \quad d_{C2} = d_{A1} * d_{B2} + d_{A2} * d_{B1},$$

$$d_{C3} = d_{A3} * d_{B1} + d_{A2} * d_{B2} + d_{A1} * d_{B3},$$

$$d_{C4} = d_{A4} * d_{B1} + d_{A3} * d_{B2} + d_{A2} * d_{B3},$$

$$d_{C5} = d_{A4} * d_{B2} + d_{A3} * d_{B3}$$

$$\gamma_1 = [(k_A + k_B) * d_{C1}]^{-1} * d_{A1} * d_{B1}[k_A * \alpha_1 + k_B * \beta_1],$$
$$\gamma_2 = [(k_A + k_B) * d_{C2}]^{-1} * [d_{A2} * d_{B1}(k_A * \alpha_2 + k_B * \beta_1) + d_{A1} * d_{B2}(k_A * \alpha_1 + k_B * \beta_2)],$$
$$\gamma_3 = [(k_A + k_B) * d_{C3}]^{-1} * [d_{A3} * d_{B1}(k_A * \alpha_3 + k_B * \beta_1) + d_{A2} * d_{B2}(k_A * \alpha_2 + k_B * \beta_2) + d_{A1} * d_{B3}(k_A * \alpha_1 + k_B * \beta_3)],$$
$$\gamma_4 = [(k_A + k_B) * d_{C4}]^{-1} * [d_{A4} * d_{B1}(k_A * \alpha_4 + k_B * \beta_1) + d_{A3} * d_{B2}(k_A * \alpha_3 + k_B * \beta_2) + d_{A2} * d_{B3}(k_A * \alpha_2 + k_B * \beta_3)],$$
$$\gamma_5 = [(k_A + k_B) * d_{C5}]^{-1} * [d_{A4} * d_{B2}(k_A * \alpha_4 + k_B * \beta_2) + d_{A3} * d_{B3}(k_A * \alpha_3 + k_B * \beta_3)],$$
$$\gamma_6 = [(k_A + k_B) * d_{C6}]^{-1} * d_{A4} * d_{B3}(k_A * \alpha_4 + k_B * \beta_3)$$

replacing the values we have:

$$[0, 0, 0.2] = 1/2 * ([0, 0, 0.2] + \beta_1),$$
$$[0.1, 0.2, 0.4] = 1/4 * [([0.2, 0.4, 0.6] + \beta_1) + ([0, 0, 0.2] + \beta_2)],$$
$$[0.28, 0.41, 0.58] = 1/6 * [([0.5, 0.7, 0.9] + \beta_1) + ([0.2, 0.4, 0.6] + \beta_2) + ([0, 0, 0.2] + \beta_3)],$$
$$[0.41, 0.58, 0.71] = 1/6 * [([0.8, 1, 1] + \beta_1) + ([0.5, 0.7, 0.9] + \beta_2) + ([0.2, 0.4, 0.6] + \beta_3)],$$
$$[0.57, 0.77, 0.87] = 1/4 * [([0.8, 1, 1] + \beta_1) + ([0.5, 0.7, 0.9] + \beta_3)],$$
$$[0.8, 1, 1] = 1/2 * ([0.8, 1, 1] + \beta_3)$$

and then:

$$\beta_1 = [0, 0, 0.2], \quad \beta_2 = [0.2, 0.4, 0.6], \quad \beta_3 = [0.8, 1, 1].$$

And finally:

$$[\alpha_4\alpha_3\alpha_2\alpha_1(k_A, d_{A4}, d_{A3}, d_{A2}, d_{A1})]\%[\gamma_6\gamma_5\gamma_4\gamma_3\gamma_2\gamma_1(k_C, d_{C6}, d_{C5}, d_{C4}, d_{C3}, d_{C2}, d_{C1})]$$
$$= \beta_3\beta_2\beta_1(k_B, d_{B3}, d_{B2}, d_{B1}) = [0.8, 1, 1][0.2, 0.4, 0.6][0, 0, 0.2](1, 1, 1, 1).$$

Because the quantities β_i and d_{Bi} are solutions of the system of equations be means of which the values d_{Ci} and γ_i are obtained, we have the following

Proposition 8.1. *Let* $[\alpha_n\alpha_{n-1}\ldots\alpha_1\ (k_A, d_{A1}, d_{A2}, \ldots, d_{An})]$ *and* $[\gamma_s\gamma_{s-1}\ldots\gamma_1\ (k_C, d_{C1}, d_{C2}, \ldots, d_{Cs})]$ *be the second parts of two attribute strings. Then*

$$[\alpha_n\alpha_{n-1}\ldots\alpha_1\ (k_A, d_{A1}, d_{A2}, \ldots, d_{An})]\%[\gamma_s\gamma_{s-1}\ldots\gamma_1\ (k_C, d_{C1}, d_{C2}, \ldots, d_{Cs})]$$
$$= [\beta_{s-n+1}\beta_{s-n}\ldots\beta_1\ (k_B, d_{B1}, d_{B2}, \ldots, d_{Bs-n+1})]$$

if and only if

$$[\alpha_n\alpha_{n-1}\dots\alpha_1\ (k_A, d_{A1}, d_{A2}, \dots, d_{An})]^\circ \times [\beta_{s-n+1}\beta_{s-n}\dots\beta_1\ (k_B, d_{B1}, d_{B2}, \dots, d_{Bs-n+1})] = [\gamma_s\gamma_{s-1}\dots\gamma_1\ (k_C, d_{C1}, d_{C2}, \dots, d_{Cs})].$$

8.3. The operation/for the first parts

We define now the operation/for the first parts:

Definition. Let $a_n a_{n-1}\dots a_1$ and $c_s c_{s-1}\dots c_1$ be the first parts of two strings A and C and suppose that the following conditions hold:
1. $c_1 \subseteq a_1$;
2. $c_s \subseteq a_n$;
3. $\bigcup_i b_i = U$.

Then if $s-n+1>0$ we define $a_n a_{n-1}\dots a_1 / c_s c_{s-1}\dots c_1 = b_m b_{m-1}\dots b_1$, where $m = s-n+1$ and

$$b_i = \left[c_i \Big\backslash \left(\bigcup_{j=0}^{i-1} b_j\right)\right] \cup \left[\bigcup_{j=i+1}^{s} (c_j \cap a_{j-i+1})\right], \quad \text{con } 1 \leqslant i \leqslant s-n+1,$$

where we consider $b_0 = \varnothing$ and $a_i = \varnothing$ for every $i > n$.

In case the previous conditions do not hold the operation is not defined.

Proposition 8.2. *The operation/is well defined.*

Proof. We have to show that $b_k \cap b_z = \varnothing$ for every $k \neq z$. One has

$$b_k = \left[c_k \Big\backslash \left(\bigcup_{j=0}^{k-1} b_j\right)\right] \cup \left[\bigcup_{j=k+1}^{s} (c_j \cap a_{j-k+1})\right]$$

and

$$b_z = \left[c_z \Big\backslash \left(\bigcup_{j=0}^{z-1} b_j\right)\right] \cup \left[\bigcup_{j=z+1}^{s} (c_j \cap a_{j-z+1})\right]$$

suppose, without loss of generality, that $k < z$; then the parts

$$\left[c_k \Big\backslash \left(\bigcup_{j=0}^{k-1} b_j\right)\right] \quad \text{and} \quad \left[c_z \Big\backslash \left(\bigcup_{j=0}^{z-1} b_j\right)\right],$$

respectively of b_k and b_z, are disjoint because such are the quantities c_i, in turn, the quantities

Proposition 8.3. *For each couple of first parts* $a_n a_{n-1} \ldots a_1$ *and* $c_s c_{s-1} \ldots c_1$. *If there is* $b_m b_{m-1} \ldots b_1$ *tale such that* $a_n a_{n-1} \ldots a_1 * b_m b_{m-1} \ldots b_1 = c_s c_{s-1} \ldots c_1$ *then one has* $a_n a_{n-1} \ldots a_1 / c_s c_{s-1} \ldots c_1 = b_m b_{m-1} \ldots b_1$.

Proof. Consider the string $c_s c_{s-1} \ldots c_1$, one has $c_1 = a_1 \cap b_1$ thus if our aim is to deduce b_1 we can just say that $b_1 \supseteq c_1$.

However, we also have $c_2 = (b_1 \cap a_2) \cup (b_2 \cap a_1)$ and from this it follows $b_1 \supseteq c_2 \cap a_2$ since in c_2 are present only elements of a_1, a_2, b_1, b_2, consequently all the elements that belong both to c_2 and a_2 belong also to b_1.

Moreover, one has $b_2 \supseteq c_2/b_1$, because all the elements of c_2 that do not belong to b_1 are to belong to $(b_2 \cap a_1)$ and thus to b_2.

The previous analysis can be summarized as follows: $b_1 \supseteq c_1 \cup (c_2 \cap a_2)$ and $b_2 \supseteq c_2 \setminus b_1$.

In a similar way one can consider $c_3 = (b_1 \cap a_3) \cup (b_2 \cap a_2) \cup (b_3 \cap a_1)$ and we get

$$\begin{aligned} &b_1 \supseteq c_1 \cup (c_2 \cap a_2) \cup (c_3 \cap a_3), \\ &b_2 \supseteq c_2 \setminus b_1 \cup (c_3 \cap a_2), \\ &b_3 \supseteq c_3 \setminus (b_1 \cup b_2). \end{aligned}$$

In such way we get the formulas of the operation/for the first parts:

$$b_i \supseteq g_i = \left[c_i \Bigg\backslash \left(\bigcup_{j=0}^{i-1} b_j \right) \right] \cup \left[\bigcup_{j=i+1}^{s} (c_j \cap a_{j-i+1}) \right],$$

where $1 \leqslant i \leqslant s - n + 1$ ($b_0 = \varnothing$ and $a_i = \varnothing$ for every $i > n$). In case $\cup g_i = U$ then we reach the conclusion that $g_i = b_i$ for every i, because additional elements cannot be added to the g_i which, in turn, represent a partition of U. □

We note that the operation/cannot always be seen as the inverse of the operation $*$. Thus the following proposition, given a couple of first parts $a_n a_{n-1} \ldots a_1$ and $c_s c_{s-1} \ldots c_1$, gives us a method to verify the existence of a string $b_m b_{m-1} \ldots b_1$ such that $a_n a_{n-1} \ldots a_1 * b_m b_{m-1} \ldots b_1 = c_s c_{s-1} \ldots c_1$.

Proposition 8.4. *For each couple of first parts* $a_n a_{n-1} \ldots a_1$ and $c_s c_{s-1} \ldots c_1$ *there is* $b_m b_{m-1} \ldots b_1$ *such that* $a_n a_{n-1} \ldots a_1 * b_m b_{m-1} \ldots b_1 = c_s c_{s-1} \ldots c_1$ *if and only if* $(a_n a_{n-1} \ldots a_1 / c_s c_{s-1} \ldots c_1)$ *is defined and* $a_n a_{n-1} \ldots a_1 * (a_n a_{n-1} \ldots a_1 / c_s c_{s-1} \ldots c_1) = c_s c_{s-1} \ldots c_1$.

Proof. If $(a_n a_{n-1} \ldots a_1 / c_s c_{s-1} \ldots c_1)$ is defined one has $a_n a_{n-1} \ldots a_1 * (a_n a_{n-1} \ldots a_1 / c_s c_{s-1} \ldots c_1) = c_s c_{s-1} \ldots c_1$, then $b_m b_{m-1} \ldots b_1 = (a_n a_{n-1} \ldots a_1 / c_s c_{s-1} \ldots c_1)$.

The inverse implication stems directly from the Proposition 8.3. □

Example. Consider the previous examples.

• When $a_2a_1 = \{a,b\}\{c,d\}$ and $c_3c_2c_1 = \{a\}\{b,c\}\{d\}$ and $a_2a_1/c_3c_2c_1 = b_2b_1 = \{c,a\}\{b,d\}$ we compute $a_2a_1 * b_2b_1$.

	{a,b}	{c,d}	*
	{c,a}	{b,d}	=
	{b}	{d}	
{a}	{c}	-	
{a}	{b,c}	{d}	

thus $a_2a_1 * b_2b_1 = c_3c_2c_1$.

• In the other example, we have $a_4a_3a_2a_1 = \{a,b\}\{c,d,e\}\{f,g\}\{h\}$ and $c_6c_5c_4c_3c_2c_1 = \{b\}\{a\}\{c\}\{e\}\{d,f,g\}\{h\}$, where $a_4a_3a_2a_1/c_6c_5c_4c_3c_2c_1 = b_3b_2b_1 = \{b\}\{a,c,d\}\{f,g,h,e\}$. We compute $a_4a_3a_2a_1 * b_3b_2b_1$:

		{a,b}	{c,d,e}	{f,g}	{h}	*
		{b}	{a,c,d}	{f,g,h,e}		
		∅	{e}	{f,g}	{h}	
	{a}	{c,d}	∅	∅	-	
{b}	∅	∅	∅	-		
{b}	{a}	{c,d}	{e}	{f,g}	{h}	

thus

$$a_4a_3a_2a_1 * b_3b_2b_1 = \{b\}\{a\}\{c,d\}\{e\}\{f,g\}\{h\} \neq c_6c_5c_4c_3c_2c_1.$$

However, if we compute $a_4a_3a_2a_1/\{b\}\{a\}\{c,d\}\{e\}\{f,g\}\{h\}$. We get: $b_1 = \{h\} \cup \{f,g\} \cup \{e\} = \{e,f,g,h\}$, $b_2 = \emptyset \cup \emptyset \cup \{c,d\} \cup \{a\} = \{a,c,d\}$ and $b_3 = \emptyset \cup \emptyset \cup \emptyset \cup \{b\} = \{b\}$. Namely the previous result, as foreseen, by the Proposition 8.3.

Now we are going to define a new operation, denoted by $\triangleright$ and we show that, provided some conditions hold, the equation $A \Diamond X = C$, has the only solution $X = A \triangleright C$.

8.4. The operation $\triangleright$ for the attribute strings

Definition. Let $A = a_n^{\alpha_n} a_{n-1}^{\alpha_{n-1}} \ldots a_2^{\alpha_2} a_1^{\alpha_1}\ (k_A, d_{A,n}, \ldots, d_{A,2}, d_{A,1})$ and $C = c_s^{\gamma_s} c_{s-1}^{\gamma_{s-1}} \ldots c_2^{\gamma_2} c_1^{\gamma_1}\ (k_C, d_{C,s}, \ldots, d_{C,2}, d_{C,1})$ be two attribute strings. Then

$$A \triangleright C = b_{s-n+1}^{\beta_{s-n+1}} b_{m-1}^{\beta_{m-1}} \ldots b_2^{\beta_2} b_1^{\beta_1}\ (k_B, d_{B,s-n+1}, \ldots, d_{B,2}, d_{B,1}).$$

If and only if $a_n a_{n-1} \dots a_1 / c_s c_{s-1} \dots c_1$ is defined, $a_n a_{n-1} \dots a_1 / c_s c_{s-1} \dots c_1 = b_m b_{m-1} \dots b_1$ and

$$[\alpha_n \alpha_{n-1} \dots \alpha_1 \ (k_A, d_{A1}, d_{A2}, \dots d_{An})]\%[\gamma_s \gamma_{s-1} \dots \gamma_1 \ (k_C, d_{C1}, d_{C2}, \dots, d_{Cs})]$$

is defined and one has

$$\begin{aligned}&[\alpha_n \alpha_{n-1} \dots \alpha_1 \ (k_A, d_{A1}, d_{A2}, \dots, d_{An})]\%[\gamma_s \gamma_{s-1} \dots \gamma_1 \ (k_C, d_{C1}, d_{C2}, \dots, d_{Cs})]\\ &\quad = [\beta_{s-n+1} \beta_{s-n} \dots \beta_1 \ (k_B, d_{B1}, d_{B2}, \dots, d_{Bs-n+1})].\end{aligned}$$

Moreover, one has

$$A \rhd U^{\mathrm{NI}} = U^{\mathrm{NI}}, \quad U^{\mathrm{NC}} \rhd A = A \quad \text{and} \quad A \rhd A = U^{\mathrm{NC}}.$$

For every string A belonging to $S(U)$. In the other cases the operation is not defined.

Proposition 8.5. *For each couple of strings A and C if there is B such that $A \lozenge B = C$ then $B = A \rhd C$.*

Proof. We note that the operation $\rhd$ arises by applying the operations / and % to the first and second parts, respectively, hence the proof follows from Propositions 8.1 and 8.3. □

Given A and C, the following proposition is very useful to verify whether a string B exists such that $A \lozenge B = C$.

Proposition 8.6. *For each couple of strings A and C there is a string B such that $A \lozenge B = C$ if and only if $A \lozenge (A \rhd C) = C$.*

Proof. Immediate from Propositions 8.1 and 8.4. □

Example. Consider

$$\begin{aligned}A &= a_4^{\alpha_4} a_3^{\alpha_3} a_2^{\alpha_2} a_1^{\alpha_1} \ (k_A, d_{A4}, d_{A3}, d_{A2}, d_{A1})\\ &= \{a,b\}^{[0.8,1,1]}\{c,d,e\}^{[0.5,0.7,0.9]}\{f,g\}^{[0.2,0.4,0.6]}\{h\}^{[0,0,0.2]}(1,1,1,1,1)\end{aligned}$$

and

$$\begin{aligned}C &= c_6^{\gamma_6} c_5^{\gamma_5} c_4^{\gamma_4} c_3^{\gamma_3} c_2^{\gamma_2} c_1^{\gamma_1} \ (k_C, d_{C6}, d_{C5}, d_{C4}, d_{C3}, d_{C2}, d_{C1})\\ &= \{b\}^{[0.8,1,1]}\{a\}^{[0.5,0.8,0.9]}\{c\}^{[0.4,0.6,0.7]}\{e\}^{[0.2,0.4,0.6]}\{d,f,g\}^{[0.1,0.2,0.4]}\{h\}^{[0,0,0.2]}\\ &\quad \times (2,1,2,3,3,2,1).\end{aligned}$$

From previous examples one has

$$[\alpha_4\alpha_3\alpha_2\alpha_1\ (k_A, d_{A4}, d_{A3}, d_{A2}, d_{A1})]\%[\gamma_6\gamma_5\gamma_4\gamma_3\gamma_2\gamma_1\ (k_C, d_{C6}, d_{C5}, d_{C4}, d_{C3}, d_{C2}, d_{C1})]$$
$$= \beta_3\beta_2\beta_1(k_B, d_{B3}, d_{B2}, d_{B1}) = [0.8, 1, 1][0.2, 0.4, 0.6][0, 0, 0.2](1, 1, 1, 1)$$

and

$$a_4a_3a_2a_1/c_6c_5c_4c_3c_2c_1 = b_3b_2b_1 = \{b\}\,\{a, c, d\}\,\{f, g, h, e\}$$

$$A \triangleright C = b_3^{\beta_3}b_2^{\beta_2}b_1^{\beta_1}\ (k_B, d_{B3}, d_{B2}, d_{B1})$$
$$= \{b\}^{[0.8,1,1]}\{a, c, d\}^{[0.2,0.4,0.6]}\{f, g, h, e\}^{[0,0,0.2]}(1, 1, 1, 1).$$

9. Further processing case study data

We remember that in case study we have obtained the following results as regards the average constitution of the boys in the sample:

$$C = \{14\}^{\mathrm{b[st]}}\{7\}^{\mathrm{ib[rs,st]}}\{8, 38\}^{\mathrm{nt[rs]}}\{4, 11\}^{\mathrm{rs}}$$
$$\times \{2, 5, 16, 18, 19, 22, 23, 27, 31\}^{\mathrm{ib[a4,rs]}}\{3, 10, 17, 21, 24, 25, 26, 35\}^{\mathrm{nt[a4]}}$$
$$\times \{12\}^{\mathrm{a4}}\{1, 13, 29, 36\}^{\mathrm{b[a4]}}\{6, 9, 15, 28, 30, 32, 34, 37\}^{\mathrm{ib[w,a4]}}\{33\}^{\mathrm{nt[w]}}$$
$$\times \{20\}^{\mathrm{w}}.$$

Suppose now that we look for the classification at age 15 such that the individual 20 might attain an average constitution better than "rather weak".

For the sake of simplicity we just take into account the subset of individuals $1, 3, 10, 14, 20$, i.e., those that most affect the classification. Let us consider the classifications $C13$, $C14$ and $C15$ as starting point, then:

$$C13 = \{14\}^{\mathrm{b[st]}}\{3\}^{\mathrm{nt[a4]}}\{10\}^{\mathrm{a4}}\{1\}^{\mathrm{b[a4]}}\{20\}^{\mathrm{w}},$$

$$C14 = \{14\}^{\mathrm{b[st]}}\{3, 10\}^{\mathrm{nt[a4]}}\{1\}^{\mathrm{a4}}\{20\}^{\mathrm{w}},$$

$$C15 = \{14\}^{\mathrm{b[st]}}\{10\}^{\mathrm{nt[a4]}}\{3, 1\}^{\mathrm{a4}}\{20\}^{\mathrm{w}}.$$

such strings can be written in full as follows:

$$C13 = \{14\}^{[0.74,0.94,0.98]}\{3\}^{[0.26,0.46,0.66]}\{10\}^{[0.2,0.4,0.6]}\{1\}^{[0.16,0.32,0.52]}$$
$$\times \{20\}^{[0,0,0.2]}(1, 1, 1, 1, 1, 1),$$

$$C14 = \{14\}^{[0.74,0.94,0.98]}\{3, 10\}^{[0.26,0.46,0.66]}\{1\}^{[0.2,0.4,0.6]}\{20\}^{[0,0,0.2]}(1, 1, 1, 1, 1),$$

$$C15 = \{14\}^{[0.74,0.94,0.98]}\{10\}^{[0.26,0.46,0.66]}\{3, 1\}^{[0.2,0.4,0.6]}\{20\}^{[0,0,0.2]}(1, 1, 1, 1, 1)$$

the composition of $C13$ and $C14$ gives

$$\begin{aligned} C13\Diamond C14 = {} & \{14\}^{[0.74,0.94,0.98]}\varnothing^{[0.5,0.7,0.82]}\{3\}^{[0.4,0.6,0.74]} \\ & \times \{10\}^{[0.32,0.49,0.65]}\varnothing^{[0.22,0.37,0.55]}\{1\}^{[0.13,0.26,0.46]}\varnothing^{[0.09,0.18,0.38]} \\ & \times \{20\}^{[0,0,0.2]}(2,1,2,3,4,4,3,2,1) \\ \approx {} & \{14\}^{\mathrm{b[st]}}\{3\}^{\mathrm{ib[a4,rs]}}\{10\}^{\mathrm{nt[a4]}}\{1\}^{\mathrm{ib[w,a4]}}\{20\}^{\mathrm{w}} \end{aligned}$$

and finally:

$$\begin{aligned} C &= C13\Diamond C14\Diamond C15 \\ &= \{14\}^{[0.74,0.94,0.98]}\varnothing^{[0.58,0.78,0.87]}\varnothing^{[0.49,0.69,0.81]}\varnothing^{[0.42,0.6,0.74]}\{3,10\}^{[0.34,0.51,0.67]} \\ &\quad \times \varnothing^{[0.27,0.43,0.60]}\varnothing^{[0.22,0.36,0.54]}\{1\}^{[0.16,0.27,0.46]}\varnothing^{[0.1,0.19,0.39]}\varnothing^{[0.06,0.12,0.32]} \\ &\quad \times \{20\}^{[0,0,0.2]}(3,1,3,6,10,13,14,13,10,6,3,1) \\ &\approx \{14\}^{\mathrm{b[st]}}\{3,10\}^{\mathrm{ib[a4,rs]}}\{1\}^{\mathrm{ib[w,a4]}}\{20\}^{\mathrm{w}}. \end{aligned}$$

Thus, if our wish is that the average constitution of individuals 20 and 1 might improve in the future, namely having the following classification:

$$\begin{aligned} C' = {} & \{14\}^{[0.74,0.94,0.98]}\varnothing^{[0.58,0.78,0.87]}\varnothing^{[0.49,0.69,0.81]}\varnothing^{[0.42,0.6,0.74]}\{3,10\}^{[0.34,0.51,0.67]} \\ & \times \varnothing^{[0.27,0.43,0.60]}\{1\}^{[0.22,0.36,0.54]}\{20\}^{[0.16,0.27,0.46]}\varnothing^{[0.1,0.19,0.39]}\varnothing^{[0.06,0.12,0.32]} \\ & \times \varnothing^{[0,0,0.2]}(3,1,3,6,10,13,14,13,10,6,3,1) \end{aligned}$$

the constitution of the boys at 15 would have to be $C'15 = (C13\Diamond C14)C'$, i.e.,

$$\begin{aligned} C'15 &= \{14,20\}^{[0.74,0.94,0.98]}\{1,10\}^{[0.26,0.46,0.66]}\{3\}^{[0.2,0.4,0.6]}\varnothing^{[0,0,0.2]}(1,1,1,1,1) \\ &\approx \{14,20\}^{\mathrm{b[st]}}\{1,10\}^{\mathrm{nt[a4]}}\{3\}^{\mathrm{a4}}. \end{aligned}$$

However, this result requires that the individual 20 *weak* at 14 would become *strong* at 15 and this fact is very unlikely. Thus we should look also for another classification at age 14, namely $C'14$. Because

$$\begin{aligned} C13\Diamond C14 = {} & \{14\}^{[0.74,0.94,0.98]}\varnothing^{[0.5,0.7,0.82]}\{3\}^{[0.4,0.6,0.74]}\{10\}^{[0.32,0.49,0.65]} \\ & \times \varnothing^{[0.22,0.37,0.55]}\{1\}^{[0.13,0.26,0.46]}\varnothing^{[0.09,0.18,0.38]} \\ & \times \{20\}^{[0,0,0.2]}(2,1,2,3,4,4,3,2,1) \end{aligned}$$

we look for $C'14$ such that

$$\begin{aligned} C13\Diamond C'14 = {} & \{14\}^{[0.74,0.94,0.98]}\varnothing^{[0.5,0.7,0.82]}\{3\}^{[0.4,0.6,0.74]}\{10\}^{[0.32,0.49,0.65]} \\ & \times \varnothing^{[0.22,0.37,0.55]}\{1\}^{[0.13,0.26,0.46]}\{20\}^{[0.09,0.18,0.38]} \\ & \times \varnothing^{[0,0,0.2]}(2,1,2,3,4,4,3,2,1) \end{aligned}$$

and we have

$$C'14 = \{14\}^{[0.74,0.94,0.98]}\{3,10\}^{[0.26,0.46,0.66]}\{1,20\}^{[0.2,0.4,0.6]}\varnothing^{[0,0,0.2]}(1,1,1,1,1)$$
$$\approx \{14\}^{\mathrm{b[st]}}\{3,10\}^{\mathrm{nt[a4]}}\{1,20\}^{\mathrm{a4}}.$$

Thus we get

$$C'15 = (C13\lozenge C'14) \rhd C$$
$$= \{14\}^{[0.74,0.94,0.98]}\{1,10,20\}^{[0.26,0.46,0.66]}\{3\}^{[0.2,0.4,0.6]}\varnothing^{[0,0,0.2]}(1,1,1,1,1)$$
$$\approx \{14\}^{\mathrm{b[st]}}\{1,10,20\}^{\mathrm{nt[a4]}}\{3\}^{\mathrm{a4}}.$$

In other terms we have tried to develop a growth plan for boys 1 and 20 in order to investigate how their average constitution can be benefically affected. Our results can be summarized in the following table:

Age	1 Actual constitution	1 Programmed constitution	20 Actual constitution	20 Programmed constitution
13	b[average constitution]	b[average constitution]	Weak	Weak
14	Average constitution	Average constitution	Weak	Average constitution
15	Average constitution	nt[average constitution]	Weak	nt[average constitution]

10. Further developments

In this section we just quote some trends for our research.

10.1. A metric space on $S(U)$

Our strings consist of not homogeneous quantities (sets and fuzzy numbers) thus it is difficult to find a function able to represent the inherent complexity of the strings. It is a hard task finding a function f such that $f(A) \neq f(B) \iff A \neq B$. It could be useful thinking to a string $A = a_n^{\alpha_n}\ldots a_2^{\alpha_2}a_1^{\alpha_1}$ as an array

$$(1/k_A, h(\alpha_n), a_n, 1/d_n, h(\alpha_{n-1}), a_{n-1}, 1/d_n, \ldots, h(a_1), \alpha_1, 1/d_1)).$$

With $h(a_i)$ we mean the number associated to the subset a_i of U, where we suppose that the function h is positive and strictly increasing with respect to the cardinality of the subsets and moreover $h(\varnothing) = 0$.

We note that in the $n+1$-ple $1/k_A$ and d_i are present and this happens because our $n+1$-ples should agree with the definition of $\leqslant$.

Now we can introduce a measure of distance for the arrays [3,22,26]. In fact, let $A = a_n^{\alpha_n} a_{n-1}^{\alpha_{n-1}} \dots a_2^{\alpha_2} a_1^{\alpha_1}$ and $B = b_m^{\beta_m} b_{m-1}^{\beta_{m-1}} \dots b_2^{\beta_2} b_1^{\beta_1}$ be a couple of strings and let $(x_1, x_2, x_3, \dots, x_{s-1}, x_s)$ and $(y_1, y_2, y_3, \dots, y_{s-1}, y_s)$ be two associated arrays, where $s = \max(2 * n + 1, 2 * m + 1)$ and the shortest string of s is "filled up" with NI instead of the label, with $h(\emptyset)$ instead of the subsets, and with 0 instead of $1/d_i$.

For example, if $A = [a, b, f, d]^{\mathrm{t}}[c, e, i]^{\mathrm{at}}[g, h]^{\mathrm{f}}(4, 2, 3, 2)$ and $B = [a, b, c, e, i]^{\mathrm{ct}}[f, d, g, h]^{\mathrm{nt[ct]}}(8, 2, 2)$, then the corresponding arrays are: (1/4, t, $h(\{a, b, f, d\})$, 1/2, at, $h(\{c, e, i\})$, 1/3, $f, h(\{g, h\})$, 1/2) and (1/8, ct, $h(\{a, b, c, e, i\})$, 1/2, nt[ct], $h(\{f, d, g, h\})$, 1/2, NI, $h(\emptyset)$).

We can also introduce a distance function on $S(U)$:

$$d(A, B) = \sum_{i=1}^{s} |x_i - y_i|,$$

where the difference between two labels $|\alpha - \beta|, (\alpha = [w_1, w_2, w_3]$ and $\beta = [z_1, z_2, z_3])$ can be interpreted as $|w_1 - z_1| + |w_2 - z_2| + |w_3 - z_3|$ and where the labels NI and NC can be viewed, respectively, as $[-1, -1, -1]$ and $[2, 2, 2]$.

10.2. The weights

Once we have generated the strings to "multiply", we obtain a resulting string which represents a new classification, considering all the attributes in the same measure. In practical situation we know that frequently the influence of an attribute on the results of the classification must be differentiated from that of the other attributes. We note that in [18,20] a weighting mechanism has been introduced, however, it is not compatible with the algebraic properties of the l-monoid and thus a new strategy should be devised.

11. Concluding remarks

In this paper, we have illustrated the basic properties of an algebraic structure and we have applied the structure to a well known case study present in the literature. Our approach can be summarized as follows:

1. A *classification*is a particular string, and more classifications can be *multiplied* by applying the operation $\Diamond$. In such way a new classification is obtained which takes into account clusters and fuzzy attributes of the initial strings.
2. A *linear ordering* is present in the structure, so that strings can be compared as regards their contents of information. Once the ordering relation is introduced the structure becomes a *limited distributive lattice*.

3. The structure includes special linguistic labels: *Not compatible* and *No information*. Introducing these labels allows that the structure be endowed with a unit element with respect to the operation $\Diamond$ and a minimum element with respect to the ordering. In such way the structure becomes an *integral commutative l-monoid*.
4. We have tackled the problem of *solving the equation* $A \Diamond X = B$, namely, given a classification B and a component A, find the other component X. This kind of problem has been rarely tackled, however it is relevant for applications.
5. Finally, the concept of *relevance of a component classification* with respect to a resulting classification and the concept of *absolute relevance* of a string have been introduced. The two measures of relevance give rise to the *total relevance*.

The methodology has been compared with the 3-way fuzzy clustering one and the results are encouraging. Our approach introduces a suitable framework in the field of soft computing and it appears as a tool flexible and powerful for data processing in fuzzy environments. We think that it is important to place the research for classification algorithms in a suitable theoretical framework.

Appendix A

	{g}	∅	{c,d}	{e,f}	{a,b}	∅	⊗
				{a,b,f}	{e,d}	{g,c}	=
		{g}	∅	{c}	∅	∅	∅
	∅	∅	{d}	{e}	∅	∅	-
∅	∅	∅	{f}	{a,b}	∅	-	
∅	∅	{g}	{d,f}	{a,b,c,e}	∅	∅	∅

the associated list is

		1	2	3	3	2	1	®
					1	1	1	=
		1	2	3	3	2	1	
	1	2	3	3	2	1	–	
1	2	3	3	2	1	–		
1	3	6	8	8	6	3	1	

thus

$$d_{\text{tall}\Diamond\text{fat}\Diamond\text{agile},1} = 1 \quad d_{\text{tall}\Diamond\text{fat}\Diamond\text{agile},2} = 3 \quad d_{\text{tall}\Diamond\text{fat}\Diamond\text{agile},3} = 6 \quad d_{\text{tall}\Diamond\text{fat}\Diamond\text{agile},4} = 8$$
$$d_{\text{tall}\Diamond\text{fat}\Diamond\text{agile},5} = 8 \quad d_{\text{tall}\Diamond\text{fat}\Diamond\text{agile},6} = 6 \quad d_{\text{tall}\Diamond\text{fat}\Diamond\text{agile},7} = 3 \quad d_{\text{tall}\Diamond\text{fat}\Diamond\text{agile},8} = 1$$

and

$$k_{\text{tall}\Diamond\text{fat}\Diamond\text{agile}} = k_{\text{tall}\Diamond\text{fat}} + k_{\text{agile}} = 2 + 1 = 3$$

for the second parts one has:

$$\alpha_6 \quad \alpha_5 \quad \alpha_4 \quad \alpha_3 \quad \alpha_2 \quad \alpha_1 \quad ©$$
$$\beta_3 \quad \beta_2 \quad \beta_1 \quad =$$

		$a_{6,1}$	$a_{5,1}$	$a_{4,1}$	$a_{3,1}$	$a_{2,1}$	$a_{1,1}$
	$a_{6,2}$	$a_{5,2}$	$a_{4,2}$	$a_{3,2}$	$a_{2,2}$	$a_{1,2}$	-
$a_{6,3}$	$a_{5,3}$	$a_{4,3}$	$a_{3,3}$	$a_{2,3}$	$a_{1,3}$	-	
γ_8	γ_7	γ_6	γ_5	γ_4	γ_3	γ_2	γ_1

$$\begin{aligned}
a_{1,1} &= d_{\text{tall}\Diamond\text{fat},1} * d_{\text{agile},1} * (k_{\text{tall}\Diamond\text{fat}} * \alpha_1 + k_{\text{agile}} * \beta_1) \\
&= 1*1*(2*[0,0,0.2]+1*[0,0,0.2]) = [0,0,0.4]+[0,0,0.2] = [0,0,0.6], \\
a_{1,2} &= 1*1*(2*[0,0,0.2]+1*[0.2,0.4,0.6]) = [0,0,0.4]+[0.2,0.4,0.6] \\
&= [0.2,0.4,1], \\
a_{1,3} &= 1*1*(2*[0,0,0.2]+1*[0.5,0.7,0.9]) = [0,0,0.4]+[0.5,0.7,0.9] \\
&= [0.5,0.7,1.3], \\
a_{2,1} &= 2*1*(2*[0.1,0.2,0.4]+1*[0,0,0.2]) = 2*([0.2,0.4,0.8]+[0,0,0.2]) \\
&= 2*[0.2,0.4,1] = [0.4,0.8,2], \\
a_{2,2} &= 2*1*(2*[0.1,0.2,0.4]+1*[0.2,0.4,0.6]) = 2*([0.2,0.4,0.8] \\
&\quad +[0.2,0.4,0.6]) = 2*[0.4,0.8,1.4] = [0.8,1.6,2.8] \\
&= 2*[0.2,0.4,1] = [0.4,0.8,2], \\
a_{2,3} &= 2*1*(2*[0.1,0.2,0.4]+1*[0.5,0.7,0.9]) = 2*([0.2,0.4,0.8] \\
&\quad +[0.5,0.7,0.9]) = 2*[0.7,1.2,1.7] = [1.4,2.4,3.4], \\
a_{3,1} &= 3*1*(2*[0.28,0.41,0.58]+1*[0,0,0.2]) = 3*([0.56,0.82,1.16] \\
&\quad +[0,0,0.2]) = 3*[0.56,0.82,1.36] = [1.68,2.46,4.08], \\
a_{3,2} &= 3*1*(2*[0.28,0.41,0.58]+1*[0.2,0.4,0.6]) = 3*([0.56,0.82,1.16] \\
&\quad +[0.2,0.4,0.6]) = 3*[0.76,1.22,1.76] = [2.28,3.66,5.28], \\
a_{3,3} &= 3*1*(2*[0.28,0.41,0.58]+1*[0.5,0.7,0.9]) = 3*([0.56,0.82,1.16] \\
&\quad +[0.5,0.7,0.9]) = 3*[1.06,1.52,2.06] = [3.18,4.56,6.18], \\
a_{4,1} &= 3*1*(2*[0.41,0.58,0.71]+1*[0,0,0.2]) = 3*([0.82,1.16,1.42] \\
&\quad +[0,0,0.2]) = 3*[0.82,1.16,1.62] = [2.46,3.48,4.86], \\
a_{4,2} &= 3*1*(2*[0.41,0.58,0.71]+1*[0.2,0.4,0.6]) = 3*([0.82,1.16,1.42] \\
&\quad +[0.2,0.4,0.6]) = 3*[1.02,1.56,2.22] = [3.06,4.68,6.66], \\
a_{4,3} &= 3*1*(2*[0.41,0.58,0.71]+1*[0.5,0.7,0.9]) = 3*([0.82,1.16,1.42] \\
&\quad +[0.5,0.7,0.9]) = 3*[1.32,1.86,2.52] = [3.96,5.58,7.56],
\end{aligned}$$

$$a_{5,1}=2*1*(2*[0.57,0.77,0.87]+1*[0,0,0.2])=2*([1.14,1.54,1.74]+[0,0,0.2])=2*[1.14,1.54,1.94]=[2.28,3.08,3.88],$$
$$a_{5,2}=2*1*(2*[0.57,0.77,0.87]+1*[0.2,0.4,0.6])=2*([1.14,1.54,1.74]+[0.2,0.4,0.6])=2*[1.34,1.94,2.34]=[2.68,3.88,4.68],$$
$$a_{5,3}=2*1*(2*[0.57,0.77,0.87]+1*[0.5,0.7,0.9])=2*([1.14,1.54,1.74]+[0.5,0.7,0.9])=2*[1.64,2.24,2.64]=[3.28,4.48,5.28],$$
$$a_{6,1}=1*1*(2*[0.8,1,1]+1*[0,0,0.2])=([1.6,2,2]+[0,0,0.2])=[1.6,2,2.2],$$
$$a_{6,2}=1*1*(2*[0.8,1,1]+1*[0.2,0.4,0.6])=([1.6,2,2]+[0.2,0.4,0.6])=[1.8,2.4,2.6],$$
$$a_{6,3}=1*1*(2*[0.8,1,1]+1*[0.5,0.7,0.9])=([1.6,2,2]+[0.5,0.7,0.9])=[2.1,2.7,2.9],$$

thus we get

$$\gamma_1=[(k_{\text{tall}\Diamond\text{fat}}+k_{\text{agile}})*d_{\text{tall}\Diamond\text{fat}\Diamond\text{agile},1}]^{-1}*a_{1,1}=3*1]^{-1}*[0,0,0.6]=(1/3)*[0,0,0.6]=[0,0,0.2]\approx\mathrm{F},$$
$$\gamma_2=(1/9)*([0.2,0.4,1]+[0.4,0.8,2])=(1/9)*[0.6,1.2,3]=[0.06,0.133,0.33]\approx\mathrm{ib[F,AV]},$$
$$\gamma_3=(1/18)*([1.68,2.46,4.08]+[0.4,0.8,2]+[0.5,0.7,1.3])=[0.14,0.22,0.41]\approx\mathrm{ib[F,AV]},$$
$$\gamma_4=(1/24)*([2.46,3.48,4.86]+[2.28,3.66,5.28]+[1.4,2.4,3.4])=[0.25,0.39,0.56]\approx\mathrm{AV},$$
$$\gamma_5=[0.355,0.54,0.73]\approx\mathrm{ib[AV,V]},$$
$$\gamma_6=[0.45,0.63,0.8]\approx\mathrm{b[V]},$$
$$\gamma_7=[0.5,0.76,0.87]\approx\mathrm{nt[V]},$$
$$\gamma_8=[0.7,0.9,0.96]\approx\mathrm{ib[V,CV]},$$

finally one has

$$\text{tall}\Diamond\text{fat}\Diamond\text{agile}=\varnothing^{\mathrm{ib[V,CV]}}\varnothing^{\mathrm{nt[V]}}\{g\}^{\mathrm{b[V]}}\{d,f\}^{\mathrm{ib[AV,V]}}\times\{a,b,c,e\}^{\mathrm{AV}}\varnothing^{\mathrm{ib[F,AV]}}\varnothing^{\mathrm{ib[F,AV]}}\varnothing^{\mathrm{F}}(3,1,3,6,8,8,6,3,1).$$

Consider

$$A=\{c,g\}^{\mathrm{CV}}d^{\mathrm{V}}\{b,e,f\}^{\mathrm{AV}}\{a\}^{\mathrm{F}},$$

and

$$B=\{g\}^{\mathrm{CV}}\{a,d,e,f\}^{\mathrm{AV}}\{b,c\}^{\mathrm{F}},$$

$$C=A\Diamond B=\{g\}^{\mathrm{CV}}\varnothing^{\mathrm{AV}}\{c,d\}^{\mathrm{ib[AV,V]}}\{e,f\}^{\mathrm{AV}}\{a,b\}^{\mathrm{ib[F,AV]}}\varnothing^{\mathrm{F}}.$$

Appendix B

$$\mu_A^C = \sum_{j=1,\ldots,4} \sum_{k=1,\ldots,6} (p_{jk} - p_j)^2,$$

$$p_1 = \#[\{a\}]/6 = 1/6,$$
$$p_2 = \#[\{b,e,f\}]/6 = 3/6 = 1/2,$$
$$p_3 = 1/6,$$
$$p_4 = 1/3,$$

$$\begin{array}{llll}
p_{1,1} = \#[\{a\} \cap \emptyset] = 0 & p_{2,1} = 0 & p_{3,1} = 0 & p_{4,1} = 0 \\
p_{1,2} = \#[\{a\} \cap \{a,b\}] = 1 & p_{2,2} = 1 & p_{3,2} = 0 & p_{4,2} = 0 \\
p_{1,3} = \#[\{a\} \cap \{e,f\}] = 0 & p_{2,3} = 2 & p_{3,3} = 0 & p_{4,3} = 0 \\
p_{1,4} = 0 & p_{2,4} = 0 & p_{3,4} = 1 & p_{4,4} = 1 \\
p_{1,5} = 0 & p_{2,5} = 0 & p_{3,5} = 0 & p_{4,5} = 0 \\
p_{1,6} = 0 & p_{2,6} = 0 & p_{3,6} = 0 & p_{4,6} = 1
\end{array}$$

thus

$$\begin{aligned}
\mu_A^C &= (0-1/6)^2 + (1-1/6)^2 + (0-1/6)^2 + (0-1/6)^2 + (0-1/6)^2 \\
&\quad + (0-1/6)^2 + (0-1/2)^2 + (1-1/2)^2 + (2-1/2)^2 + (0-1/2)^2 \\
&\quad + (0-1/2)^2 + (0-1/2)^2 + (0-1/6)^2 + (0-1/6)^2 + (0-1/6)^2 \\
&\quad + (1-1/6)^2 + (0-1/6)^2 + (0-1/6)^2 + (0-1/3)^2 + (0-1/3)^2 \\
&\quad + (0-1/3)^2 + (1-1/3)^2 + (0-1/3)^2 + (1-1/3)^2 \\
&= 1/36 + 25/36 + 1/36 + 1/36 + 1/36 + 1/36 + 1/4 + 1/4 + 9/4 \\
&\quad + 1/4 + 1/4 + 1/4 + 1/36 + 1/36 + 1/36 + 25/36 + 1/36 + 1/36 \\
&\quad + 1/9 + 1/9 + 1/9 + 4/9 + 1/9 + 4/9 \\
&= 5/6 + 7/2 + 5/6 + 4/3 = 6.5
\end{aligned}$$

while $\mu_B^C = 5.5$.

$$p_1 = 1/3, \quad p_2 = 2/3, \quad p_3 = 1/6,$$

$$\begin{array}{lll}
p_{1,1} = 0 & p_{2,1} = 0 & p_{3,1} = 0 \\
p_{1,2} = 1 & p_{2,2} = 1 & p_{3,2} = 0 \\
p_{1,3} = 0 & p_{2,3} = 2 & p_{3,3} = 0 \\
p_{1,4} = 1 & p_{2,4} = 1 & p_{3,4} = 0 \\
p_{1,5} = 0 & p_{2,5} = 0 & p_{3,5} = 0 \\
p_{1,6} = 0 & p_{2,6} = 0 & p_{3,6} = 1
\end{array}$$

thus

$$\begin{aligned}\mu_B^C &= (0-1/3)^2+(1-1/3)^2+(0-1/3)^2+(1-1/3)^2+(0-1/3)^2\\ &\quad+(0-1/3)^2+(0-2/3)^2+(1-2/3)^2+(2-2/3)^2+(1-2/3)^2\\ &\quad+(0-2/3)^2+(0-2/3)^2+(0-1/6)^2+(0-1/6)^2+(0-1/6)^2\\ &\quad+(0-1/6)^2+(0-1/6)^2+(1-1/6)^2\\ &=1/9+4/9+1/9+4/9+1/9+1/9+4/9+1/9+16/9+1/9\\ &\quad+4/9+4/9+1/36+1/36+1/36+1/36+1/36+25/36\\ &=4/3+10/3+5/6=5.5.\end{aligned}$$

Appendix C

$$\begin{aligned}\rho_B &:= \sum_{i=1,\dots,n-1}\ \sum_{j=i+1,\dots,n} |\alpha_i-\alpha_j|\\ &= (|0-0.2|+|0-0.4|+|0.2-0.6|)+(|0-0.8|+|0-1|+|0.2-1|)\\ &\quad+(|0.2-0.8|+|0.4-1|+|0.6-1|)\\ &=1+2.6+1.6=5.2\end{aligned}$$

thus

$$\frac{(3*2)*4}{2*5.2}=2.3$$

and $\rho_B=(2/\pi)\,\mathrm{arctg}(2.3)=0.73$.

Finally one has $R_B^C=\mu_B^C*\rho_B=5.5*0.73=4.01$.

Appendix D

(i) Consider the string $A=\{b,d\}^{[0.8,0.9,1]}\{a,e,c\}^{[0.5,0.6,0.7]}\{f\}^{[0.4,0.5,0.6]}(2,1,2,1)$ resulting from the composition of other strings.

We get

$$\begin{aligned}\sum_{i=1,\dots,n-1}\ \sum_{j=i+1,\dots,n} |\alpha_i-\alpha_j| &= (|0.6-0.7|+|0.5-0.6|+|0.4-0.5|)\\ &\quad+(|0.6-1|+|0.5-0.9|+|0.4-0.8|)\\ &\quad+(|0.7-1|+|0.6-0.9|+|0.5-0.8|)\\ &=0.3+1.2+0.9=2.4\end{aligned}$$

Appendix B

$$\mu_A^C = \sum_{j=1,\dots,4} \sum_{k=1,\dots,6} (p_{jk} - p_j)^2,$$

$$p_1 = \#[\{a\}]/6 = 1/6,$$
$$p_2 = \#[\{b,e,f\}]/6 = 3/6 = 1/2,$$
$$p_3 = 1/6,$$
$$p_4 = 1/3,$$

$$\begin{array}{llll}
p_{1,1} = \#[\{a\} \cap \varnothing] = 0 & p_{2,1} = 0 & p_{3,1} = 0 & p_{4,1} = 0 \\
p_{1,2} = \#[\{a\} \cap \{a,b\}] = 1 & p_{2,2} = 1 & p_{3,2} = 0 & p_{4,2} = 0 \\
p_{1,3} = \#[\{a\} \cap \{e,f\}] = 0 & p_{2,3} = 2 & p_{3,3} = 0 & p_{4,3} = 0 \\
p_{1,4} = 0 & p_{2,4} = 0 & p_{3,4} = 1 & p_{4,4} = 1 \\
p_{1,5} = 0 & p_{2,5} = 0 & p_{3,5} = 0 & p_{4,5} = 0 \\
p_{1,6} = 0 & p_{2,6} = 0 & p_{3,6} = 0 & p_{4,6} = 1
\end{array}$$

thus

$$\begin{aligned}
\mu_A^C = {} & (0-1/6)^2 + (1-1/6)^2 + (0-1/6)^2 + (0-1/6)^2 + (0-1/6)^2 \\
& + (0-1/6)^2 + (0-1/2)^2 + (1-1/2)^2 + (2-1/2)^2 + (0-1/2)^2 \\
& + (0-1/2)^2 + (0-1/2)^2 + (0-1/6)^2 + (0-1/6)^2 + (0-1/6)^2 \\
& + (1-1/6)^2 + (0-1/6)^2 + (0-1/6)^2 + (0-1/3)^2 + (0-1/3)^2 \\
& + (0-1/3)^2 + (1-1/3)^2 + (0-1/3)^2 + (1-1/3)^2 \\
= {} & 1/36 + 25/36 + 1/36 + 1/36 + 1/36 + 1/36 + 1/4 + 1/4 + 9/4 \\
& + 1/4 + 1/4 + 1/4 + 1/36 + 1/36 + 1/36 + 25/36 + 1/36 + 1/36 \\
& + 1/9 + 1/9 + 1/9 + 4/9 + 1/9 + 4/9 \\
= {} & 5/6 + 7/2 + 5/6 + 4/3 = 6.5
\end{aligned}$$

while $\mu_B^C = 5.5$.

$$p_1 = 1/3, \quad p_2 = 2/3, \quad p_3 = 1/6,$$

$$\begin{array}{lll}
p_{1,1} = 0 & p_{2,1} = 0 & p_{3,1} = 0 \\
p_{1,2} = 1 & p_{2,2} = 1 & p_{3,2} = 0 \\
p_{1,3} = 0 & p_{2,3} = 2 & p_{3,3} = 0 \\
p_{1,4} = 1 & p_{2,4} = 1 & p_{3,4} = 0 \\
p_{1,5} = 0 & p_{2,5} = 0 & p_{3,5} = 0 \\
p_{1,6} = 0 & p_{2,6} = 0 & p_{3,6} = 1
\end{array}$$

thus

$$\frac{(3*2)*(2+1+2+1)}{2*2.4}=\frac{36}{4.8}=7.5$$

and $\rho_A = (2/\pi) * \mathrm{arctg}(7.5) = 0.915$.

Consider now $A = \{b,d\}^{[0.6,0.65,0.7]}\{a,e,c\}^{[0.5,0.6,0.7]}\{f\}^{[0.4,0.5,0.6]}(4,1,2,1)$. We have

$$\begin{aligned}\sum_{i=1,\dots,n-1}\sum_{j=i+1,\dots,n}|\alpha_i-\alpha_j| &= (|0.6-0.7|+|0.5-0.6|+|0.4-0.5|)\\ &\quad+(|0.6-0.7|+|0.5-0.65|+|0.4-0.6|)\\ &\quad+(|0.7-0.7|+|0.6-0.65|+|0.5-0.6|)\\ &=0.3+0.45+0.15=0.9\end{aligned}$$

thus

$$\frac{(3*2)*(4+1+2+1)}{2*0.9}=\frac{48}{1.8}=26.6$$

and $\rho_A = (2/\pi) * \mathrm{arctg}(26.6) = 0.97$.

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