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An algebraic fuzzy structure for approximate reasoning

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Abstract: The concept of C-set and of the related C-calculus was introduced to study some standard problems of signal processing. Now, it is shown how the theory of fuzzy sets can lead to the C-fsets and to the C-fcalculus. Afterwards, we focus our attention on the theory of approximate reasoning and we show how C-fcalculus can be regarded, in some cases, as a possible alternative to the theory of approximate reasoning.

Keywords: Fuzzy sets; C-fsets; C-fcalculus; approximate reasoning.

1. Introduction

C-sets and C-calculus were originally proposed to study some problems of pattern analysis and processing of signals [1, 2]. The main idea in C-calculus is to consider the two distinct roles played by the figures in a number, i.e. value and position, and to use them for defining appropriate 'composite' sets (C-sets, for short) and related operations.

The theory of approximate reasoning (AR) was, in turn, developed by Zadeh [7] to provide a tool for handling information which is fuzzy and non-specific. Linguistic statements are translated into possibility distributions and the latter are then suitably manipulated by means of various rules which represent the reasoning mechanism of the system.

Our main purpose here, after some basic definitions and results about C-sets are briefly recalled, is to bring fuzziness into C-calculus and to investigate the relationships between C-fcalculus and the theory of AR.

2. C-Calculus

The study of hierarchical systems led to the introduction of C-sets as an answer to the need of characterizing in a 'natural' way some basic features of these systems. We recall some basic definitions and properties.

(1) Let $\alpha_n, \alpha_{n-1}, \ldots, \alpha_0$ be some covering of a given set U, then the corresponding *C-set* is described by the string $\alpha_n\, \alpha_{n-1} \cdots \alpha_0$.

(2) The basic operations of sum and product among C-sets are defined like the corresponding arithmetical operations, with the difference that the sum (product) of digits takes the place of the union (intersection) of sets.

In the following we will use the symbols $\oplus$ and $\otimes$ to denote the set-theoretic union and intersection, respectively. Furthermore, the application to digits of $\oplus$ and $\otimes$ will be meant as the 'corresponding' lattice-theoretic operations l.u.b. and g.l.b. It can be easily shown that both the sum and the product are commutative. Moreover, the distributivity of the product with respect to the sum allows us to claim that the collection of all C-sets defined in U constitutes a *commutative semi-ring* with respect to sum

Definition. Let A be an arbitrary subset of R. A is a *fuzzy number* if and only if:
(a) A is normalized and convex;
(b) there is a unique $x_0 \in R$ such that $\mu_A(x_0) = 1$;
(c) μ is stepwise continuous.

Definition. We can accomplish a hierarchy of fuzzy sets via 'fuzzification'. More precisely we get fuzzy sets of the *second type* if $\mu_A(x)$ is itself a fuzzy set. The meaning is that the grade of membership cannot be precisely associated to each element. Moreover, we have fuzzy set of the *second level* when the elements of the sets are themselves fuzzy sets.

Fuzzy logic [6] deals with truth values which are fuzzy numbers belonging to the interval [0, 1] and where each value has a *linguistic label.*

Fuzzy logic can be considered as a fuzzy extension of multi-valued logic and it is based on fuzzy sets of the second type. In order to define a fuzzy logic one has first to define the set of truth values and then to obtain, via inference rules, from fuzzy statements other fuzzy ones.

The theory of AR [7] is based on fuzzy logics: we recall some basic features of the theory and the related idea of possibility theory. The basic propositions in the theory of AR are statements of the form

$$p \equiv x \text{ is } A$$

where x is a variable and A is some subset (possibly fuzzy) of the universe of discourse T.

A possibility distribution $\Pi_V : X \to [0, 1]$ is induced by the above statement, such that for each $x \in X$,

$$\Pi_V(x) = A(x)$$

where $\Pi_V(x)$ is to be viewed as the possibility of V being x and $A(x)$ is the membership grade of x in A.

For example, consider the statement

$P \equiv$ Mark is tall;

the information conveyed by the statement can be represented in the form

height(Mark) = tall.

Next consider the proposition,

P: Mark is much taller than John.

We can express this knowledge in the form

(height(Mark), height(John)) = much taller than.

Thus, the theory of AR essentially aims at reducing possibilities, i.e. if x is a variable with universe of discourse X, as we obtain more information the possibility distribution is reduced from its initial value of $\Pi_V(x) = 1$. Then if we lack knowledge about a variable we have $\Pi_V(x) = 1$ for all x in its universe.

4. C-Calculus and fuzzy sets

Our goal is to extend the theory of C-calculus by considering fuzzy sets instead of ordinary sets. We remember that a lattice structure was superimposed on C-sets; thus, first, we try to introduce a partial ordering among fuzzy sets. We note that, in order to define an ordering among ordinary sets, one has to single out some properties of their elements, considered as equivalent with respect to the characterizing properties.

When we deal with fuzzy sets, their elements have different membership functions so that the ordering relation should be defined directly among each element. Moreover, the fuzzy relation should measure to what extent the elements share the relation. However, we note that C-sets are described by strings, a linear structure which allows only binary logic.

Proof. Immediate since C-calculus already characterizes a semi-ring and moreover the fuzzy union and intersection verify the semi-ring structural properties. Finally, we note that -if the characteristic functions of the fuzzy sets are considered, we get the ordinary sets.

4.2. C-fsets of the second type

In this case we have fuzzy numbers associated with ordinary sets and our problem is to assign a possibility distribution, i.e. a fuzzy number, to each element. There are no more set-theoretic operations; we have to extend the algebraic operations to the fuzzy numbers. We choose to take the maximum and the minimum among fuzzy numbers so that the result is still a fuzzy number. In such way it is trivial to show that the family of C-fsets of the second type also constitutes a commutative semi-ring.

5. An application of C-fcalculus to the approximate reasoning

In this section we aim to show how C-fcalculus can supply an inferential process and a knowledge representation comparable with approximate reasoning and which, in particular cases, can work satisfactorily. The statements object of study in the theory of AR will be represented via C-fsets and then the operations of C-fcalculus will be utilized in order to obtain an inference rule.

First, we note that fuzzy logic is based on fuzzy sets of the second type so that C-fsets of the second type are to be considered as 'natural' candidates.

Moreover, if we consider nested fuzzy statements, of the form

$(u \text{ is } A) \text{ is } \tau$

where A is a unary fuzzy relation and τ is a truth value, we can devise a C-fset with an appropriate ordering among the truth values τ.

In fact, we can collect all the elements having the same value of τ, i.e. satisfying A to the same extent. If we suppose that the values are comparable, we get an ordering relation among the sets corresponding to each value τ and the resulting string of fuzzy sets represents a C-fset.

More precisely, we suppose that the truth values are quantities which can be associated with a fuzzy number. Moreover, we note that the operations $\oplus$ and $\otimes$ correspond, in fuzzy logic, to the operations 'and' and 'or', respectively. The operation 'not' can be implemented by complementing the fuzzy numbers corresponding to the sets present in the C-fset.

However, the essential difference between C-fcalculus and fuzzy logic is the inherent parallelism of C-fcalculus. For example, if we consider two C-fsets, the first associated with a statement A and the second with B, by applying the operation we get a new C-fset and a new statement $(A \oplus B)$.

An important remark is the following. In the theory of AR we start from some statements and, via the extant relations, we get other statements in a transitive way: we mean that if x_1 verifies the relation A and the relation B holds between x_1 and x_2, we can deduce how x_2 satisfies A.

Our approach is entirely different: we have a family of fuzzy statements each characterizing a unary relation among the elements of the universe. C-fcalculus still supports a deductive logic but we emphasize that the latter is an associative one.

In fact, if x_1 satisfies the relation A with grade τ_1 and the relation B with grade τ_2, we infer that x_1 satisfies C with grade τ_3 where C is the statement deduced by A and B and τ_3 is obtained by connecting τ_1 and τ_2.

Appropriate application fields of C-fcalculus are situations in which fuzzy items are present and these can be obtained having other fuzzy items as points of departure. In fact, we can associate with each item a C-fset and by applying the above operations we can infer the corresponding results.

Example. Consider a population whose elements suffer, to different extents, from, say, 'hypertension', 'diabetes', 'cholesterol'. Moreover we have to consider also the social and psychological behavior of the individuals since these symptoms are as important as the previous ones in order to give rise to the where

$$\tau_1 = \text{at},$$
$$\tau_2 = \text{LA}[\text{qt} \cap \text{at}],$$
$$\tau_3 = \text{LA}[\text{qt} \cup (\text{t} \cap \text{at})],$$
$$\tau_4 = \text{LA}[(\text{vt} \cap \text{at}) \cup (\text{qt} \cap \text{t})],$$
$$\tau_5 = \text{LA}[(\text{vt} \cap \text{qt}) \cup \text{t}],$$
$$\tau_6 = \text{LA}[\text{vt} \cap \text{t}],$$
$$\tau_7 = \text{vt},$$

where LA denotes the Linguistic Approximation. The resulting C-fset will be

$$D_1 = A_7^\tau \cdots A_1^\tau.$$

Finally we apply the operation $\otimes$ to the C-fsets D_1 and D_2:

$$D_1 \otimes D_2 = M$$

and the C-fset M, associated with the disease coronaropathy is finally obtained; M contains ten sets. In M the population is just clustered in terms of 'coronaropathy'. The individuals that share the highest fuzzy numbers in M are the most likely candidates to the disease.

6. Concluding remarks

The notion of C-fset provides, in our opinion, a convenient point of departure for the building of a conceptual framework which parallels in many respects that utilized in the case of fuzzy logic and approximate reasoning, yet its generality, potentially, may induce a wide scope of applicability. C-fCalculus provides a 'natural' way of dealing with problems where fuzzy items are deducible from other fuzzy items.

Applications of this nature are being deeply investigated and research in this direction is well under way.

References

[1] E.R. Caianiello, A. Gisolfi and S. Vitulano, C-Calculus an overview, in: *Cybernetic System: Recognition Learning Self-Organization* (John Press, Italia, 1982).

[2] E.R. Caianiello, A. Gisolfi and S. Vitulano, A technique for texture analysis using C-calculus, in: *Signal Processing* 1 (North-Holland, Amsterdam, 1979) 159–173.

[3] A. Gisolfi and S. Vitulano, C-matrix, C-filter: an application to human chromosomes, in: R.M. Haralick, Ed., *Pictorial Data Analysis, Series F: Computer and Systems Sciences*, 4 (Springer-Verlag, Berlin, 1983) 69–85.

[4] A. Gisolfi and S. Vitulano, Algebraic pattern recognition, in: V. Cappellini and A.G. Constantinides, Eds., *Digital Signal Processing* (North-Holland, Amsterdam, 1984) 738–750.

[5] L.A. Zadeh, Fuzzy sets, *Inform. and Control* **8** (1965) 338–353.

[6] L.A. Zadeh, Fuzzy logic and approximate reasoning, *Synthese* **30** (1975) 407–428.

[7] L.A. Zadeh, A theory of approximate reasoning, in: J.E. Hayes, M. Michie and L.I. Kulich, Eds., *Machine Intelligence*, 9 (Wiley, New York, 1979).