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AN ALGEBRAIC APPROACH TO THE APPROXIMATE REASONING

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ABSTRACT

The notion of C-set was originally introduced to cope with some standard problems such as texture analysis and feature extraction. Now, it is shown how the strands of C-calculus and fuzzy systems can be woven to lead to the C-fcalculus. Then we show that the C-fcalculus can be regarded, in some cases, as a possible alternative to the theory of approximate reasoning. The main advantages of our approach can be summarised as follows: data compression, immediate comparisons among the elements of the universe, parallel processing of the elements, logical deduction of associative type. Finally the Prolog implementation running on PC's is briefly discussed.

INTRODUCTION

The theory of approximate reasoning (AR) was developed by Zadeh (7) in order to furnish an adequate tol for handling information which is non-specific and fuzzy. We transform, according to the theory, linguistic statements into possibility distributions and then the rules which represent the reasoning mechanism of the system are applied to such distribution. Formally the theory of AR is based on fuzzy logics (6) which can be considered as a fuzzy extension of the multi-valued logics and are based on fuzzy sets of second type (5).

The basic statements in the theory of AR are propositions fo the form: $P \equiv x$ is A where x is a variable and A is some subset (possibly fuzzy) of the universe of discourse. For example, consider the statement P = John is tall , we can represent the information conveyed by the statement as follows:

height (John) = tall.

The point of departure, in turn, of the C-calculus are the two distinct roles played by the figures in a number, i.e. value and position, and to use them for defining suitable *composite* sets (*C-sets* for short) and related operations (1).

<u>Definition</u>. 1) Let $\alpha_n, \alpha_{n-1}, \ldots, \alpha_0$ be some covering of a given set U, then the corresponding *C-set* is described by the string $\alpha_n \alpha_{n-1} \ldots \alpha_0$

2) The basic operations of sum and product among C-sets are defined like the corresponding arithmetical operations, with the difference that the sum (product) of digits takes the place of the union (intersection) of sets.

It has been shown that the collection of all C-sets defined in U constitutes a commutative semi-ring with respect to sum and product. Moreover a suitable relation of partial ordering was introduced so that the resulting structure was a lattice (4). A family of C-sets can be easily associated with a digitized picture and some standard problems, e.g. texture analysis and feature extraction, have been tackled by introducing appropriate algorithms (2,3).

Our main purpose here is to take fuzziness into the C-calculus so that *C-fsets* can be appropriately defined and furthermore to investigate the relationships between the *C-fcalculus* and the theory of AR.

C-CALCULUS AND FUZZY SETS.

First we try to introduce a partial ordering among fuzzy sets: we note that their elements have different membership functions so that the ordering relation should be defined directly among each element. Furthermore, the fuzzy relation should measure to what extent the relation itself is satisfied by the elements. Then, our plan should be accomplished in two steps:

a) ordering of the elements belonging to a fuzzy set by means of a relation among their membership functions; b) ordering of the fuzzy sets belonging to a C-set.

This plan can be carried out via a suitable "labeling" of each set: "linguistic" labels should be considered as natural candidates since fuzzy sets are associated with concepts that can be expressed by linguistic forms. For instance, one could associate a C-set with some characteristic of its elements and thus each fuzzy sets should collect all of the elements satisfying, to some extent, the feature itself.

The above discussion and the circumstance that the fuzzy logic is based on fuzzy sets of second type lead us to the following

Definition. Let $\alpha_n, \alpha_{n-1}, \ldots, \alpha_0$ be a family of fuzzy sets in X, then the corresponding *C-fset* of second type is described by the string whose elements are the fuzzy numbers corresponding to each fuzzy set.

Thus a C-fset of second type is a string of fuzzy numbers associated with each fuzzy set and the ordering relation holds among the fuzzy numbers. Consider now what kind of operations can be allowed to the C-fsets. Beyond refining the information contained in the C-fsets one has to evaluate new membership functions associated with the original ones.

Fuzzy numbers are associated with ordinary sets and we have to assign a possibility distribution, i.e. a fuzzy number, to each element. In order to extend the algebraic operations to the fuzzy numbers we choose to take the maximum and the minimum among fuzzy numbers so that the resulting number is still fuzzy. Then it is easily shown that the family of all C-fsets of second type in X, together with these two operations, constitutes a commutative semi-ring. Thus we have a generalization of the C-calculus that reduces to it in case of ordinary sets.

C-FCALCULUS AND THE APPROXIMATE REASONING

In our approach the statements objects of study in the theory of AR will be represented by C-fsets and then the operations of the C-fcalculus are applied to obtain an appropriate inference rule. Thus we show how the C-fcalculus can provide a knowledge representation and inferential processes comparable with the approximate reasoning and such that, in particular cases, can perform satisfactorily.

First we note that a C-fset can be considered as a couple of C-fsets: the former is called *primary* and contains ordinary sets, the latter is called *secondary* and is formed by fuzzy numbers. Then we have to define two types of operations associated with the couple of C-fsets. More precisely, for the primary C-fset, we will use the symbols $\oplus$ and $\otimes$ to denote the set-theoretic union and intersection, respectively. The same symbols, in turn, when referred to the secondary one, denote union and intersection of fuzzy numbers.

Let us consider fuzzy statements of the form (u is A) is τ where A is an unary fuzzy relation and τ is a truth value, then we can construct a C-fset having an appropriate ordering among the values τ. In fact, all the elements sharing the same value τ, i.e. satisfying A to the same extent, can be collected. Then assuming that the values are comparable an ordering relation among the sets associated with each value τ is gotten and thus the resulting string of fuzzy sets represents a C-fset as follows:

$C = \alpha_n^{\tau_n} \alpha_{n-1}^{\tau_{n-1}} \ldots \alpha_0^{\tau_0}$ where the $\alpha_n, \alpha_{n-1}, \ldots, \alpha_0$ are the sets having the same τ_i. In such way we get immediate comparisons among the elements of the universe, data compression, easy reading of the data. However it is worth emphasizing the inherent parallelism of the C-fcalculus which allows parallel processing of the elements.

Example. We observe that the operations $\oplus$ and $\otimes$ correspond, in the fuzzy logic, to the operations AND and OR, respectively. Then let us apply the operation $\otimes$ to the C-fsets $C_1 = \alpha_3^{\tau_3} \alpha_2^{\tau_2} \alpha_1^{\tau_1}$ and $C_2 = \beta_2^{\xi_2} \beta_1^{\xi_1}$ considering separately the sets and the fuzzy numbers.

$$\alpha_3 \quad \alpha_2 \quad \alpha_1 \quad \otimes$$

$$\beta_2 \quad \beta_1 \quad =$$

$$\beta_1 \cap \alpha_3 \quad \beta_1 \cap \alpha_2 \quad \beta_1 \cap \alpha_1$$

$$\beta_2 \cap \alpha_3 \quad \beta_2 \cap \alpha_2 \quad \beta_2 \cap \alpha_1$$

$$\beta_2 \cap \alpha_3 \quad [(\beta_1 \cap \alpha_3) \cup (\beta_2 \cap \alpha_2)] \quad [(\beta_1 \cap \alpha_2) \cup (\beta_2 \cap \alpha_1)] \quad \beta_1 \cap \alpha_1$$

and

$$
\begin{array}{cccc}
\tau_3 & \tau_2 & \tau_1 & \otimes \\
 & \xi_2 & \xi_1 & = \\
\hline
 & \xi_1 \cap \tau_3 & \xi_1 \cap \tau_2 & \xi_1 \cap \tau_1 \\
\xi_2 \cap \tau_3 & \xi_2 \cap \tau_2 & \xi_2 \cap \tau_1 & \\
\hline
\xi_2 \cap \tau_3 & [(\xi_1 \cap \tau_3) \cup (\xi_2 \cap \tau_2)] & [(\xi_1 \cap \tau_2) \cup (\xi_2 \cap \tau_1)] & \xi_1 \cap \tau_1
\end{array}
$$

The resulting C-fset is $C_3 = v_4^{\mu_4} v_3^{\mu_3} v_2^{\mu_2} v_1^{\mu_1}$ and represents a finer description of the universe of discourse.

Thus a new C-fset is gotten by applying the operation $\otimes$ and then we have to define the corresponding statement by considering the statements associated with the original C-fsets:

(u is A) is $\tau_1 \otimes$ (u is B) is ξ_1 = (u is LA [A $\otimes$ B] is LA $[\tau_1 \otimes \xi_1]$;

(u is C) is δ_1

where "LA" denotes the "linguistic approximation". Thus we start from fuzzy statements and get new fuzzy ones. We have an approximate inferential process, yet an important remark is the following: in the theory of AR we start from some statements and, via the extant relations, we get other statements in a transitive way, i.e. if x_1 verifies the relation A and the relation B holds between x_1 and x_2, we can deduce in what manner x_2 satisfies A. Our approach is utterly different: in fact, we have some relations among the elements of the universe and each of them is characterized by a fuzzy statement. The C-fcalculus still supports a deductive logic but we stress that the latter is now an associative one. In fact, if the relations A and B are satisfied by x_1 with grade τ_1 and ξ_1, respectively, we can infer that x_1 satisfies C with grade δ_1 where C is the statement deduced by A and B and δ_1 is gotten by connecting τ_1 and ξ_1

Suitable application fields of the C-fcalculus are the situations in which fuzzy items are present and the latter can be obtained starting from other fuzzy items. In fact, a C-fset can be associated with each item and then the corresponding results are achieved by applying the operations.

AN APPLICATION

Consider a population which should be classified according to the "old age". We construct as many C-fsets as are the distinguishing symptoms (e.g. white hair, slowness, bad sight, toothless). A C-fset contains the individuals that suffer from a certain symptom and a fuzzy number (i.e. a linguistic label) is associated with each of them. Of course the labels have to be the same for all the C-fsets so that the operations can be carried out consistently. We apply the operation $\oplus$ for symptoms considered as similar; in turn, the symptoms inherently dishomogenous need the operation $\otimes$. For the sake of simplicity we consider only four grades: "false", "almost true", "true", "very true". In the following we write, for short, "f", "at", "t", "vt". Thus the symptom "white hair" gives rise to the C-fset

$C_1 = \alpha_4^{vt} \alpha_3^{t} \alpha_2^{at} \alpha_1^{f}$

and the same happens for the other three symptoms:

$C_2 = \beta_4^{vt} \beta_3^{t} \beta_2^{at} \beta_1^{f}$ $\quad C_3 = \delta_4^{vt} \delta_3^{t} \delta_2^{at} \delta_1^{f}$ $\quad C_4 = \mu_4^{vt} \mu_3^{t} \mu_2^{at} \mu_1^{f}$

As the four symptoms are not similar then the operation $\otimes$ has to be applied in order to obtain the resulting C-fset. First of all, we have to consider the sets α_i and β_i and carry out the well-known set-theoretic intersection which gives rise to a sequence of seven sets. Let us consider instead what happens to the fuzzy numbers.

$vt \quad t \quad at \quad f \quad \otimes \quad vt \quad t \quad at \quad f \quad =$

			$vt\cap f$	$t\cap f$	$at\cap f$	$f\cap f$
		$vt\cap at$	$t\cap at$	$at\cap at$	$f\cap at$	
	$vt\cap t$	$t\cap t$	$at\cap t$	$f\cap t$		
$vt\cap vt$	$t\cap vt$	$at\cap vt$	$f\cap vt$			
τ_7	τ_6	τ_5	τ_4	τ_3	τ_2	τ_1

where $\tau_1=f$; $\tau_2=LA[at\cap f]$; $\tau_3=LA[(t\cap f)\cup at]$; $\tau_4=LA[(t\cap at)\cup(f\cap vt)]$; $\tau_5=LA[(vt\cap at)\cup t)]$; $\tau_6=LA[vt\cap t]$; $\tau_7=vt$.

Thus the resulting C-fset is $D_1 = A_7^{\tau_7} A_6^{\tau_6} \ldots\ldots A_1^{\tau_1}$. For example, the set A_7 contains the individuals whose whole hair is white and are very slow. The other sets contain the individuals who suffer, to a lower degree, from this couple of symptoms. Now we apply the operation $\otimes$ to the C-fsets C_3 and C_4 and a C-fset having seven elements is gotten again: $D_2 = B_7^{\tau_7} B_6^{\tau_6} \ldots\ldots B_1^{\tau_1}$

The set B_7 contains the individuals completely toothless and suffering very much from bad sight. Finally, the product of D_1 and D_2 is carried out and the C-fset $V = E_{13}^{\tau_{13}} E_{12}^{\tau_{12}} \ldots\ldots E_1^{\tau_1}$ associated with the "old age" is gotten. In V the population is just clustered in terms of old age. It is apparent that the individuals in V having the highest fuzzy numbers, as described by the linguistic labels, are the oldest in the population.

IMPLEMENTATION

We have realized an implementation of the aforementioned algebraic structure by means of the Prolog language running on PC's having MS-DOS as operating system. The user is provided with a friendly interface which permits to carry out the data entry via a pseudo-natural language.

To represent the C-fsets we have used clauses of the form

Cfset (C, i,α_1 ,τ_1); Cfset (C, i,α_2 ,τ_2); .... Cfset (C, i,α_n ,τ_n)

where the first argument is the distinguishing feature. In a similar way every fuzzy number is represented. As classification problems generally require that many data have to be managed, a couple of hash tables (for sets and fuzzy numbers) have been introduced.

Our project involves the development and testing of a prototype devoted to medical diagnoses that we feel has a great potential since the C-fcalculus provides a "natural" way of handling problems where fuzzy items should be deduced from other ones. Our work is geared toward both exploring a conceptual framework which strictly parallels that utilized in the case of approximate reasoning and validating the approach in terms of practical effectiveness.

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